

Dessins d'enfants: bipartite maps and Galois groups

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Abstract. Belyĭ's Theorem implies that the Riemann surfaces defined over the field of algebraic numbers are precisely those which support bipartite maps; this provides a faithful representation of the Galois group of this field on these combinatorial objects.

My aim in this note is to show how combinatorics can play a central role in uniting such topics as Galois theory, algebraic number theory, Riemann surfaces, group theory and hyperbolic geometry. The relevant combinatorial objects are maps on surfaces, often called *dessins d'enfants* in view of the rather naïve appearance of some of the most common examples. For simplicity I will restrict attention to bipartite maps, though triangulations and hypermaps also play an important role in this theory. More detailed surveys can be found in [2, 5, 7], and for recent progress see [8].

A *bipartite map* \mathcal{B} consists of a bipartite graph \mathcal{G} imbedded (without crossings) in a compact, connected, oriented surface X , so that the faces (connected components of $X \setminus \mathcal{G}$) are simply connected. One can describe \mathcal{B} by a pair of permutations g_0 and g_1 of its edge-set E : the vertices can be coloured black or white, so that each edge joins a black and a white vertex; the orientation of X then determines a cyclic ordering of the edges around each black or white vertex, and these are the disjoint cycles of g_0 and g_1 respectively. These two permutations generate a subgroup $G = \langle g_0, g_1 \rangle$ of the symmetric group S^E of all permutations of E , called the *monodromy group* of \mathcal{B} ; the topological hypotheses imply that \mathcal{G} has to be connected, so G acts transitively on E . Conversely, every 2-generator transitive group arises in this way from some bipartite map \mathcal{B} : the edges are the symbols permuted, the black and white vertices correspond to the cycles of the two generators g_0 and g_1 , and the faces correspond to the cycles of $g_\infty = (g_0g_1)^{-1}$. Isomorphism of maps (preserving orientation and vertex-colours) corresponds to conjugacy of pairs (g_0, g_1) in S^E , and the automorphism group of \mathcal{B} can be identified with the centraliser of G in S^E , that is, the group of all permutations which commute with G .

(Historical note: A slight modification of these ideas allows one to describe any oriented map, whether bipartite or not, by a pair of permutations [6]. Although generally regarded as a modern development, this use of permutations can be traced back at least as far as Hamilton's construction of what we now call Hamiltonian cycles in the icosahedron [4].)

Every bipartite map \mathcal{B} is a quotient of the *universal bipartite map* $\hat{\mathcal{B}}$, drawn on the upper half-plane $\mathcal{U} = \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$; the vertices of $\hat{\mathcal{B}}$ are the rational numbers r/s (in reduced form) with s odd, coloured black or white

as r is even or odd, and the edges are the hyperbolic geodesics (euclidean semi-circles) joining vertices r/s and x/y with $ry - sx = \pm 1$. The automorphisms of $\hat{\mathcal{B}}$ are the Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbf{Z}, ad - bc = 1) \quad (*)$$

such that $a \equiv d \equiv 1$ and $b \equiv c \equiv 0 \pmod{2}$; these form a normal subgroup $\Gamma(2)$ of index 6 in the modular group $\Gamma = PSL_2(\mathbf{Z})$ of all transformations $(*)$, called the *principal congruence subgroup* of level 2 in Γ . Now $\Gamma(2)$ is a free group of rank 2, generated by the transformations

$$T_0 : z \mapsto \frac{z}{-2z + 1} \quad \text{and} \quad T_1 : z \mapsto \frac{z - 2}{2z - 3},$$

so there is an epimorphism $\theta : \Gamma(2) \rightarrow G$, $T_i \mapsto g_i$, which gives a transitive permutation representation of $\Gamma(2)$ on E . If B denotes the subgroup $\theta^{-1}(G_e)$ of $\Gamma(2)$ fixing an edge e of \mathcal{B} then B acts as a group of automorphisms of $\hat{\mathcal{B}}$, and one can show that $\mathcal{B} \cong \hat{\mathcal{B}}/B$. The underlying surface of $\hat{\mathcal{B}}/B$ is now a compact Riemann surface $\overline{\mathcal{U}/B} = (\mathcal{U} \cup \mathbf{Q} \cup \{\infty\})/B$, formed by compactifying \mathcal{U}/B with finitely many points, corresponding to the orbits of B on the extended rationals $\mathbf{Q} \cup \{\infty\}$. One can regard $\hat{\mathcal{B}}/B$ as a rigid, conformal model of \mathcal{B} , with a complex structure induced from that of \mathcal{U} : for instance, the edges are geodesics, the angles between edges around any vertex are equal, and the automorphisms of \mathcal{B} are conformal isometries of the Riemann surface.

Riemann showed that a Riemann surface X is compact if and only if it is isomorphic to the Riemann surface of an algebraic curve $f(x, y) = 0$ for some polynomial $f(x, y) \in \mathbf{C}[x, y]$. Computationally and theoretically, the most satisfactory polynomials are those with coefficients in the field $\overline{\mathbf{Q}}$ of algebraic numbers; results of Belyĭ [1] and Weil [9] imply that the Riemann surfaces corresponding to such polynomials $f(x, y) \in \overline{\mathbf{Q}}[x, y]$ are those obtained from bipartite maps by the above method. More precisely, Belyĭ showed that a compact Riemann surface X is defined over $\overline{\mathbf{Q}}$ if and only if there is a Belyĭ function β from X to the Riemann sphere $\Sigma = \mathbf{C} \cup \{\infty\}$, that is, a meromorphic function on X which is unbranched over $\Sigma \setminus \{0, 1, \infty\}$. In these circumstances, X is the underlying surface of the bipartite map $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$, where \mathcal{B}_1 is the trivial bipartite map $\hat{\mathcal{B}}/\Gamma(2)$ on Σ with a black vertex at 0, a white vertex at 1, and a single edge along the unit interval $[0, 1]$. If each edge of \mathcal{B} is identified with the sheet of the covering $\beta : X \rightarrow \Sigma$ which contains it, then the monodromy group G of \mathcal{B} coincides with the monodromy group of β , regarded as a group of permutations of the sheets; in particular, the elements $g_0, g_1, g_\infty \in G$ describe how the sheets are permuted by lifting small loops in Σ around the branch-points 0, 1 and ∞ . Similarly, the automorphism group of \mathcal{B} is identified with the group of covering transformations of β .

Since $\overline{\mathbf{Q}}$ is the union of the Galois (finite normal) extensions $K \geq \mathbf{Q}$ in \mathbf{C} , it follows that the *absolute Galois group* $\mathbf{G} = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of $\overline{\mathbf{Q}}$ over \mathbf{Q} is

the projective limit of the finite Galois groups $\text{Gal}(K/\mathbf{Q})$ of these algebraic number fields; as such, it is an uncountable profinite group, which embodies the whole of classical Galois theory over \mathbf{Q} . This group \mathbf{G} is of fundamental importance in several areas of mathematics: for instance, the representation theory of \mathbf{G} played a crucial role in Wiles's proof of Fermat's Last Theorem [10], and the Inverse Galois Problem (Hilbert's still unproved conjecture that every finite group is a Galois group over \mathbf{Q}) is equivalent to showing that every finite group is an epimorphic image of \mathbf{G} . Fortunately, Belyi's Theorem provides us with an explicit realisation of \mathbf{G} in terms of bipartite maps, which is beginning to add to our rather meagre knowledge of this complicated group.

In [3], Grothendieck showed that the natural action of \mathbf{G} on polynomials over $\overline{\mathbf{Q}}$ induces an action of \mathbf{G} on bipartite maps (and on other similar combinatorial objects, generally known as *dessins d'enfants*), through the above correspondence between maps and polynomials. Although \mathbf{G} preserves such properties of a map as its genus, the numbers and valencies of its black and white vertices, its monodromy group and its automorphism group, this action of \mathbf{G} is nevertheless faithful, in the sense that each non-identity element of \mathbf{G} sends some bipartite map to a non-isomorphic bipartite map. Moreover, this action remains faithful even when restricted to such simple objects as plane trees (maps on the sphere with one face). One therefore has a combinatorial approach to Galois theory, which is attracting interest from a wide spectrum of mathematicians and theoretical physicists (for whom maps are an effective discrete approximation to the compact Riemann surfaces which play a major role in quantum gravity).

References

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