Lecture 1

by Prof. Jürgen Wolfart

notes by Tuomas Puurtinen

1 Riemann Surfaces and Algebraic Curves

Riemann surfaces are Haudorff spaces with a countable base topology, where chart maps to $\mathbb{C}$ are defined biholomorphically where they coincide. We are discussing here only about connected Riemann surfaces.

1.1 Examples

1. Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$: Take two charts U_1 and U_2, for example $U_1 \cong \mathbb{C}$ and $U_2 \cong (\mathbb{C} - \{0\}) \cup \{\infty\}$. Then $z \mapsto \frac{1}{z}$ is a holomorphic mapping between the charts.

2. $F_n^{\text{aff}} = \{ (x, y) \in \mathbb{C}^2 \,|\, x^n + y^n = 1 \}$, $n > 1$ is a "Fermat curve". Take as charts for example $(x, y) \mapsto y$, which is homeomorphic in suitable neighboughoods of all points except $x = 0$, $y^n = 1$, and $(x, y) \mapsto x$ which is homeomorphic in suitable neighbourhoods of all points except $y = 0$, $x^n = 1$, with transition functions $x = \sqrt[n]{1 - y^n}$ and $y = \sqrt[n]{1 - x^n}$.

3. More general: all "smooth" affine algebraic curves

$$X^{\text{aff}} := \{ (x, y) \in \mathbb{C}^2 \,|\, f(x, y) = 0 \}$$

for some polynomial f with the property that in all $p \in X^{\text{aff}}$

$$\frac{\partial f}{\partial x}(p) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(p) \neq 0.$$

The implicit function theorem says that locally around p all solutions of $f(x, y) = 0$ are of the shape $(h(y), y)$ or $(x, g(x))$ where h and g are holomorphic. Then the projections serve as charts.

4. Affine hyperelliptic curves: $y^2 = (x - a_1) \cdot \ldots \cdot (x - a_n)$ with pairwise distinct $a_1, \ldots, a_n$. For example $f = y^2 - \Pi(x - a_i)$ and we have

$$\frac{\partial f}{\partial y}(p) = 2y = 0$$

in all $(a_i, 0)$, but

$$\frac{\partial f}{\partial x}(p) \neq 0$$

in these $(a_i, 0)$.

Theorem 1.1 *Let X, Y be connected Riemann surfaces, $f : X \to Y$ non-constant holomorphic mapping, $p \in X$, $f(p) = p' \in Y$. Then there exist charts $z : U(p) \to V \subset \mathbb{C}$ and $w : U'(p') \to V' \subset \mathbb{C}$ with $z(p) = 0$, $w(p') = 0$ such that*

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & U' \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle w} \\ \mathbb{C} : z & \longmapsto & w = z^n : \mathbb{C} \end{array}$$

is commutative for some choice $n \in \mathbb{N}$ independent of the choice of the charts. Constant $n = \operatorname{mult}_p f$ is the multiplicity of f in p.

If $n = 1$ then f is locally biholomorphic ("unramified at p") otherwise "ramified" of order n.

1.2 Important consequences

1. If $f : X \to \hat{\mathbb{C}}$ is meromorphic, then if it's non-constant, then the zeros and poles are discrete in X.
2. Ramification points of f are discrete in X.
3. Identity theorem, maximum principle, open mapping theorem are valid.
4. On compact Riemann surfaces we have for $f : X \to \hat{\mathbb{C}}$ (meromorphic, non-constant function) only a finite number of zeros or poles, and also a finite number of ramification points. Holomorphic functions $f : X \to \mathbb{C}$ have to be constant.
5. If $f : X \to Y$ is holomorphic and non-constant, X compact, then f is surjective and Y is compact as well.
6. Under the same hypothesis $\deg f := \sum_{p \in f^{-1}(y)} \operatorname{mult}_p f$ is independent of $y \in Y$.

1.3 More examples

1. Fermat curve:

$$F_n := \{\, [x,y,z] \in \mathbb{P}^2(\mathbb{C}) \mid x^n + y^n = z^n \,\}$$

better (symmetric) $x^n + y^n + z^n = 0$, covered by affine curves with $z = 1$, $y = 1$, $x = 1$. We'll get different affine Fermat curves, when we take charts of the form $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$. This's a typical example of a "smooth projective algebraic curve". Of cource, because $\mathbb{P}^2(\mathbb{C})$ is compact, then F_n is compact.

2. Try to compactify also hyperellictic curves

$$y^2 = \Pi_{i=1}^n (x - a_i) \Rightarrow y^2 z^{n-2} = \Pi_i (x - a_i z).$$

Now if $z = 1$ then we'll have an affine curve, or if $z = 0$ then $x^n = 0$ and so on $x = 0$, normalizing by $y = 1$ we get affine equation

$$z^{n-2} = \Pi(x - a_i).$$

Implicit function theorem isn't applicable in situations $n > 3$. Here, write $y^2 = q(x)$ with

$$\deg q = \begin{cases} 2g+1, \\ 2g+2 \end{cases}$$

and with

$$z := \frac{1}{x} \quad \text{and} \quad w := \frac{y}{x^{g+1}}.$$

$$k(z) := z^{2g+2} q\left(\frac{1}{z}\right) \in \mathbb{C}[z]$$

with $\deg k = 2g + 2$. So then $y^2 = q(x) \Leftrightarrow w^2 = k(z)$ in all points $x \neq 0, \neq z \ldots$.

$$x = y = \infty \Leftrightarrow z = 0 \Leftrightarrow \begin{cases} w = 0, & \text{for } \deg q = 2q+1 \\ w = \pm\sqrt{k(0)} & \text{for } \deg q = 2q+2 \end{cases}$$

1.4 Fact

Riemann surfaces are orientable! That is an implication from that the transition functions are biholomorphic respecting the orientation. Surfaces can also be triangulated: Suppose you have X triangulated. Then we have the Euler characteristic

$$\chi(X) := f - e + v,$$

with f, e and v counting "faces", "edges" and "vertices", which does not depend on the triangulation. For example $\chi(\hat{\mathbb{C}}) = 2$ and $\chi(\text{Torus}) = 0$.

Theorem 1.2 (Riemann-Hurwitz) *If $f : X \to Y$ is non-constant holomorphic mapping of compact Riemann surfaces, then*

$$2g(X) - 2 = \deg f(2g(Y) - 2) + \sum_{p \in X} (\mathrm{mult}_p f - 1).$$

Several applications, e.q: $g(Y) \leq g(X)$ with "=" only if f is isomorphism (unramified) or $g = 1$ and f is unramified. If $g(X) > g(Y) = 0$ or 1, then f is ramified.

Now F_n (n>2) has genus $\frac{(n-1)(n-2)}{2}$, and we consider $f : F_n \to \hat{\mathbb{C}} : [x, y, z] \mapsto \frac{x}{z}$: For example $z = 1$, $x^n + y^n = 1$, $f : (x, y) \mapsto x$ and $\deg f = n$. Exceptionally, f has only one preimage in points with $x^n = 1$: n points with $\mathrm{mult}_p f = n$. If then $z = 0$, $x^n + y^n = 0$, $y = 1$, n points on F_n, f has n poles and it's unramified. Now Riemann-Hurwitz implies that

$$2g(F_n) - 2 = n(-2) + n(n-1) = n^2 - 3n.$$

1.5 Problems

1. Find as many automorphisms of F_n as possible! (if possible, find $6n^2$ automorphisms.) Determine the structure of $\mathrm{Aut}\, F_n$.

2. Apply Riemann-Hurwitz to determine the genus of the (compact) hyperelliptic curves.

Theorem 1.3 *There is an equivalence between the categories "compact Riemann surfaces" and "smooth projective algebraic curves".*