Lecture 10

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4 Moduli Fields and Fields of Definition

"Existence of Belyi function $\beta \Rightarrow X$ is defined over $\bar{\mathbb{Q}}$."

K is a *field of definition* for the compact Riemann surface X iff X is isomorphic to a smooth projective algebraic curve $\subset \mathbb{P}^N(\mathbb{C})$ given by equations $p_i(x_0, \ldots, x_N) = 0$, all $p_i \in K[x_0, \ldots, x_N]$. If K is a field of definition, then $\mathbb{C} \supset L \supset K$ is a field of definition. Is there a minimal field of definition? Is it in $\bar{\mathbb{Q}}$?

Let

$$\underline{\underline{G}}_{\mathbb{C}} := \text{group of field automorphisms of } \mathbb{C}.$$

Suppose X to be defined over K by equations $p_i(x_0, \ldots, x_N) = 0$, take $\sigma \in \underline{\underline{G}}_{\mathbb{C}}$, let X^σ be defined by the equations $p_i^\sigma(x_0, \ldots, x_N) = 0$ (apply σ to all coefficients of all p_i) $\Leftrightarrow$

$$\{ [\sigma(x_0), \ldots, \sigma(x_N)] \mid [x_0, \ldots, x_N] \in X \} =: X^\sigma .$$

This is again a smooth curve!

By the same reason

$$\begin{array}{ccc} X & \rightsquigarrow & X^\sigma \\ \downarrow \beta & & \downarrow \beta^\sigma \\ \hat{\mathbb{C}} & \xrightarrow{\cong} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

commutes. Here β^σ is defined by applying σ to the coefficients of β, and it remains a Belyi function on X^σ, because vanishing of derivatives is preserved under σ, and $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(\infty) = \infty$. The list of all multiplicities of β is preserved under σ and degree of β equals to degree of β^σ. This implies that σ maps the dessin D for β a dessin D^σ of the same type for β^σ on X^σ. Now $\underline{\underline{G}}_{\mathbb{C}}$ acts on dessins of a given type and with a given no. of edges! Finite orbits $\Rightarrow$

Theorem 4.1 *a) The subgroup $\underline{\underline{G}}(D) := \{ \sigma \in \underline{\underline{G}}_{\mathbb{C}} \mid D \cong D^\sigma \}$ is of finite index in $\underline{\underline{G}}_{\mathbb{C}}$.*

b) $\sigma \in \underline{\underline{G}}(D) \Leftrightarrow$ *there exist (biholomorphic!) isomorphisms* $f_\sigma : X \to X^\sigma$ *for which*

$$\begin{array}{ccc} X & \xrightarrow{f_\sigma} & X^\sigma \\ {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle\beta^\sigma} & \\ \hat{\mathbb{C}} & & \end{array}$$

commutes with the respective Belyi functions.

c) The "moduli field" $M(D) := \{\, \zeta \in \mathbb{C} \,|\, \sigma(\zeta) = \zeta \ \forall \sigma \in \underline{\underline{G}}(D) \,\}$ *has finite degree* $[M(D) : \mathbb{Q}] \Rightarrow$ *is a numberfield. (Reason: all* $\zeta \in M(D)$ *have a finite orbit under* $\underline{\underline{G}}_{\mathbb{C}}$*, length of orbit is bounded by* $(\underline{\underline{G}}_{\mathbb{C}} : \underline{\underline{G}}(D)) \Rightarrow [M(D) : \mathbb{Q}] \leq (\underline{\underline{G}}_{\mathbb{C}} : \underline{\underline{G}}(D))$*.)*

Consequence: Also $\underline{\underline{G}}(X) := \{\, \sigma \in \underline{\underline{G}}_{\mathbb{C}} \,|\, \exists$ isomorph. $f_\sigma : X \to X^\sigma \,\}$ and it follows that the corresponding fixed field $M(X)$ of $\underline{\underline{G}}(X)$ in $\subseteq M(D)$, and therefore we have again a number field.

Theorem 4.2 *$M(X)$ depends only on the isomorphism class of X and is contained in any field of definition for X (analoguous for $M(D) \subset$ field of definitions for X and β).*

Suppose $X \cong X'$, i.e. there is an isomorphism $h : X \to X'$ and suppose that $\sigma \in \underline{\underline{G}}(X)$, i.e. there is an isomorphism $f_\sigma : X \to X^\sigma$. We have an isomorphism $h^\sigma : X^\sigma \to X'^\sigma$ and we can contruct an isomorphism to make the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle f_\sigma}\downarrow & & \vdots \\ X^\sigma & \xrightarrow{h^\sigma} & X'^\sigma \end{array}$$

commute. $h^\sigma \circ f_\sigma \circ h^{-1}$ gives the isomorphism we are looking for $\Rightarrow \sigma \in \underline{\underline{G}}(X')$ $\Rightarrow$

$$\underline{\underline{G}}(X) \cong \underline{\underline{G}}(X') \Rightarrow \text{ claim.}$$

Theorem 4.3 *$M(X)$ is a field of definition for X if $g(X) = 0$ or 1.*

Proof. $g = 0 \Leftrightarrow X \cong \hat{\mathbb{C}} \cong \mathbb{P}^1(\mathbb{C})$, defined $/\mathbb{Q}$.

$g(X) = 1 \Leftrightarrow X \cong \Lambda \backslash \mathbb{C}, \Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ $(\tau \in \mathbb{H})$. X defined $/\mathbb{Q}(g_2(\tau), g_3(\tau)) \supseteq \mathbb{Q}(j(\tau))$, so we see that X can be defined even over $\mathbb{Q}(j(\tau))$. X^σ defined $/\mathbb{Q}(\sigma(g_2(\tau)), \sigma(g_3(\tau)))$, even over $\mathbb{Q}(\sigma(j(\tau)))$, where $\sigma \in \underline{\underline{G}}_{\mathbb{C}}$. So $X \cong X^\sigma \Leftrightarrow \sigma(j(\tau)) = j(\tau)$. $M(X)$ is generated by $j(\tau)$ and $\mathbb{Q}(j(\tau))$ is a field of definition. □

But: in high genera there are counter examples, where X cannot be defined over $M(X)$. (Earle 1969, Shimura, Dèbes/Emsalem, Fuertes/Gonzalez).

Example by Earle in $g = 2$, $\zeta = \zeta_3 = e^{\frac{2\pi i}{3}}$

$$X : y^2 = x(x - \zeta)(x + \zeta)(x - \zeta^2 t)(x + \frac{\zeta^2}{t})$$

defined over $\mathbb{Q}(\zeta)$, and where $t \in \mathbb{Q}, t \neq 0, 1, t > 0$.

1. X cannot by defined over $\mathbb{Q}$. Note that point pairs $(\infty, \infty), (0, 0)$, $(\zeta, 0), (-\zeta, 0)$,$(\zeta^2 t, 0), (-\frac{\zeta^2}{t}, 0)$ are "intrinsic", also their image points in $\mathbb{P}^1(\mathbb{C})$ under $(x, y) \mapsto x$, upto $\mathrm{PSL}_2\,\mathbb{C}$-transformations. If X can be defined over $\mathbb{Q}$, then there is an anticonformal automorphism of X, permuting the critical points on $\mathbb{P}^1(\mathbb{C})$: doesn't exist (by calculation of cross-ratios)!

2. $M(X) = \mathbb{Q} = \mathbb{R} \cap \mathbb{Q}(\zeta)$, $X \cong \bar{X}$, i.e. there is a holomorphic isomorphism $X \to \bar{X}$. There is an anticonformal mapping $(x, y) \mapsto (-\frac{1}{\bar{x}}, \frac{i\bar{y}}{\bar{x}^3})$, which is in fact an anticonformal automorphism (of order 4).

Theorem 4.4 *If $M(X) \in \bar{\mathbb{Q}}$, then X can be defined over a number field. (Weil, J.W., B. Köck)*

Idea: Any field of definition K for X is finitely generated over $M(X)$ because for a model of X defined over K

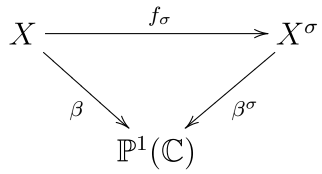
$$\sigma|_K = \mathrm{id} \Rightarrow X = X^\sigma.$$

Suppose for simplicity $K = M(X)(\xi)$ where ξ is transcendential, then there exists $\sigma \in \underline{\underline{G}}_{\mathbb{C}}$, $\sigma|_{M(X)} = \mathrm{id}_{M(X)}$, $\sigma(\xi) \mapsto \eta$ (η any other transcendential number). And because $\sigma \in \underline{\underline{G}}(X)$, there exists $f_\sigma : X \to X^\sigma$. Equations $p_i(x) = 0 \rightsquigarrow p_i^\sigma(x) = 0$ coefficients rational in $\xi \rightsquigarrow$ coefficients rational in η. Now try to insert in f_σ instead of η some algebraic $\alpha \in \bar{\mathbb{Q}}$, and it can be shown that f_σ is still an isomorphism for infinitely many $\alpha \in \bar{\mathbb{Q}}$. This gives the claim.

Theorem 4.5 (Weil) *Let X be defined over a finite extension L of $M := M(X)$. Then X can be defined over M itself if and only if $\forall \sigma \in \mathrm{Gal}\,\bar{M}/M$ there is an isomorphism $f_\sigma : X \to X^\sigma$ such that $\forall \sigma, \tau \in \mathrm{Gal}\,\bar{M}/M$ we have*

$$f_{\sigma\tau} = f_\sigma^\tau \circ f_\tau.$$

Analoguous statement holds for $M(D)$ and the field of definition for X and β, with diagram

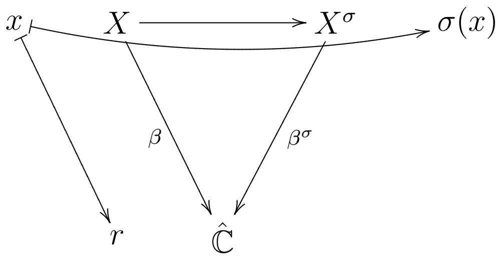


commuting.

Consequence: If $\operatorname{Aut} X = \{\mathrm{id}\}$, then X is defined over $M(X)$. $\Leftarrow$ f_σ is unique (generic case for $g > 2$).

Theorem 4.6 (Coombes/Harbater, Dèbes/Emsalem, J.W., B. Köck) *Quasiplatonic curve X can be defined $/M(X)$.*

Idea: The canonical projection $X \to \operatorname{Aut} X \backslash X \cong \mathbb{P}^1(\mathbb{C})$ is a Belyi function, assume that the critical points are $0, 1, \infty$. Let D be the corresponding (regular) dessin on X, $M(D) \subset \bar{\mathbb{Q}}$. Prove first that X, β are defined $/M(D)$. Let $r \neq 0, 1$, $r \in M(D) \subset \mathbb{C} \subset \hat{\mathbb{C}}$, fix one $x \in \beta^{-1}(r), \sigma \in \operatorname{Gal} \bar{\mathbb{Q}}/M(D)$ to make the following diagram



commute. $\sigma(r) = r \Rightarrow \sigma(x) \in (\beta^\sigma)^{-1}(r)$, choose f_σ so that $f_\sigma(x) = \sigma(x)$ $\Rightarrow$ unique choice for f_σ and it has been shown that Weil's conditions are satisfied! The proof that X can be defined even over $M(X) \subseteq M(D)$ needs some additional arguments.

4.1 Problem

7 Show that the elliptic curve can be defined over $\mathbb{Q}(j(\tau)) \subseteq \mathbb{Q}(g_2(\tau), g_3(\tau))$.

8 Suppose X defined $/\bar{\mathbb{Q}}$, $g(X) > 1$. ($\operatorname{Aut} X$ finite $\Rightarrow$) Show that all automorphisms $f : X \to X$ are also defined $/\bar{\mathbb{Q}}$.