4 Moduli Fields and Fields of Definition

"Existence of Belyi function $\beta \Rightarrow X$ is defined over $\bar{\mathbb{Q}}$.

$K$ is a field of definition for the compact Riemann surface $X$ iff $X$ is isomorphic to a smooth projective algebraic curve $\subset \mathbb{P}^N(\mathbb{C})$ given by equations $p_i(x_0, \ldots, x_N) = 0$, all $p_i \in K[x_0, \ldots, x_N]$. If $K$ is a field of definition, then $\mathbb{C} \supset L \supset K$ is a field of definition. Is there a minimal field of definition? Is it in $\bar{\mathbb{Q}}$?

Let $G_C := \text{group of field automorphisms of } \mathbb{C}$.

Suppose $X$ to be defined over $K$ by equations $p_i(x_0, \ldots, x_N) = 0$, take $\sigma \in G_C$, let $X^\sigma$ be defined by the equations $p_i^\sigma(x_0, \ldots, x_N) = 0$ (apply $\sigma$ to all coefficients of all $p_i$) $\iff$

$$\{ [\sigma(x_0), \ldots, \sigma(x_N)] | [x_0, \ldots, x_N] \in X \} =: X^\sigma.$$ 

This is again a smooth curve!

By the same reason

$$\begin{array}{ccc}
X & \xrightarrow{\sim} & X^\sigma \\
\downarrow{\beta} & & \downarrow{\beta^\sigma} \\
\mathbb{C} & \xrightarrow{\cong} & \mathbb{P}^1(\mathbb{C})
\end{array}$$

commutes. Here $\beta^\sigma$ is defined by applying $\sigma$ to the coefficients of $\beta$, and it remains a Belyi function on $X^\sigma$, because vanishing of derivatives is preserved under $\sigma$, and $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(\infty) = \infty$. The list of all multiplicities of $\beta$ is preserved under $\sigma$ and degree of $\beta$ equals to degree of $\beta^\sigma$. This implies that $\sigma$ maps the dessin $D$ for $\beta$ a dessin $D^\sigma$ of the same type for $\beta^\sigma$ on $X^\sigma$.

Now $G_C$ acts on dessins of a given type and with a given no. of edges! Finite orbits $\Rightarrow$

**Theorem 4.1**  
a) The subgroup $G(D) := \{ \sigma \in G_C | D \cong D^\sigma \}$ is of finite index in $G_C$.
b) \( \sigma \in \mathbb{G}(D) \iff \) there exist (biholomorphic!) isomorphisms \( f_{\sigma} : X \to X^\sigma \)
for which
\[
\begin{array}{ccc}
X & \xrightarrow{f_{\sigma}} & X^\sigma \\
\downarrow{\beta} & & \uparrow{\beta^\sigma} \\
\hat{\mathbb{C}}
\end{array}
\]
commutes with the respective Belyi functions.

c) The "moduli field" \( M(D) := \{ \zeta \in \mathbb{C} | \sigma(\zeta) = \zeta \ \forall \ \sigma \in \mathbb{G}(D) \} \) has finite
degree \([M(D) : \mathbb{Q}] \Rightarrow \) is a numberfield. (Reason: all \( \zeta \in M(D) \) have
a finite orbit under \( \mathbb{G}_C \), length of orbit is bounded by \( (\mathbb{G}_C : \mathbb{G}(D)) \Rightarrow \)
\([M(D) : \mathbb{Q}] \leq (\mathbb{G}_C : \mathbb{G}(D)).\)

Consequence: Also \( \mathbb{G}(X) := \{ \sigma \in \mathbb{G}_C | \exists \ \text{isomorph.} \ f_{\sigma} : X \to X^\sigma \} \) and
it follows that the corresponding fixed field \( M(X) \) of \( \mathbb{G}(X) \) in \( \subseteq M(D) \), and
therefore we have again a number field.

**Theorem 4.2** \( M(X) \) depends only on the isomorphism class of \( X \) and is
contained in any field of definition for \( X \) (analogous for \( M(D) \subset \text{field of}
definitions for } X \) and \( \beta \)).

Suppose \( X \cong X' \), i.e. there is an isomorphism \( h : X \to X' \) and suppose
that \( \sigma \in \mathbb{G}(X) \), i.e. there is an isomorphism \( f_{\sigma} : X \to X^\sigma \). We have an
isomorphism \( h^\sigma : X^\sigma \to X'^{\sigma} \) and we can contruct an isomorphism to make
the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow{f_{\sigma}} & & \uparrow{h^\sigma} \\
X^\sigma & \xrightarrow{h^\sigma} & X'^{\sigma}
\end{array}
\]
commute. \( h^\sigma \circ f_{\sigma} \circ h^{-1} \) gives the isomorphism we are looking for \( \Rightarrow \sigma \in \mathbb{G}(X') \)
\( \Rightarrow \)
\( \mathbb{G}(X) \cong \mathbb{G}(X') \Rightarrow \) claim.

**Theorem 4.3** \( M(X) \) is a field of definition for \( X \) if \( g(X) = 0 \) or 1.

**Proof.** \( g = 0 \iff X \cong \hat{\mathbb{C}} \cong \mathbb{P}^1(\mathbb{C}), \) defined \(/\mathbb{Q}.
\( g(X) = 1 \iff X \cong \Lambda \setminus \mathbb{C}, \Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau \ (\tau \in \mathbb{H}). \) X defined \(/\mathbb{Q}(g_2(\tau), g_3(\tau)) \supseteq \mathbb{Q}(j(\tau)), \) so we see that \( X \) can be defined even over \( \mathbb{Q}(j(\tau)). \) \( X^\sigma \) defined
\(/\mathbb{Q}(\sigma(g_2(\tau)), \sigma(g_3(\tau))), \) even over \( \mathbb{Q}(\sigma(j(\tau))), \) where \( \sigma \in \mathbb{G}_C. \) So \( X \cong X^\sigma \iff \sigma(j(\tau)) = j(\tau). \) \( M(X) \) is generated by \( j(\tau) \) and \( \mathbb{Q}(j(\tau)) \) is a field of definition. \[\square\]
But: in high genera there are counter examples, where $X$ cannot be defined over $M(X)$. (Earle 1969, Shimura, Dèbes/Emsalem, Fuertes/Gonzalez).

Example by Earle in $g = 2$, $\zeta = e^{2\pi i/3}:
\begin{align*}
X : y^2 = x(x - \zeta)(x + \zeta)(x - \zeta^2 t)(x + \frac{\zeta^2}{t})
\end{align*}
defined over $\mathbb{Q}(\zeta)$, and where $t \in \mathbb{Q}$, $t \neq 0, 1, t > 0$.

1. $X$ cannot be defined over $\mathbb{Q}$. Note that point pairs $(\infty, \infty), (0, 0), (-\zeta, 0), (\zeta^2 t, 0), (-\frac{\zeta^2}{t}, 0)$ are "intrinsic", also their image points in $\mathbb{P}^1(\mathbb{C})$ under $(x, y) \mapsto x$, up to $\text{PSL}_2 \mathbb{C}$-transformations. If $X$ can be defined over $\mathbb{Q}$, then there is an anticonformal automorphism of $X$, permuting the critical points on $\mathbb{P}^1(\mathbb{C})$: doesn’t exist (by calculation of cross-ratios)!

2. $M(X) = \mathbb{Q} = \mathbb{R} \cap \mathbb{Q}(\zeta)$, $X \cong \bar{X}$, i.e. there is a holomorphic isomorphism $X \to \bar{X}$. There is an anticonformal mapping $(x, y) \mapsto (-\frac{1}{x}, \frac{y}{x^3})$, which is in fact an anticonformal automorphism (of order 4).

**Theorem 4.4** If $M(X) \in \bar{\mathbb{Q}}$, then $X$ can be defined over a number field. (Weil, J.W., B. Köck)

Idea: Any field of definition $K$ for $X$ is finitely generated over $M(X)$ because for a model of $X$ defined over $K$
\[\sigma|_K = \text{id} \Rightarrow X = X^\sigma.\]

Suppose for simplicity $K = M(X)(\xi)$ where $\xi$ is transcendental, then there exists $\sigma \in G_{\infty}$, $\sigma|_{M(X)} = \text{id}_{M(X)}$, $\sigma(\xi) \mapsto \eta$ ($\eta$ any other transcendental number). And because $\sigma \in G(X)$, there exists $f_\sigma : X \to X^\sigma$. Equations $p_i(x) = 0 \sim p_i^\sigma(x) = 0$ coefficients rational in $\xi \sim \eta$. Now try to insert in $f_\sigma$ instead of $\eta$ some algebraic $\alpha \in \bar{\mathbb{Q}}$, and it can be shown that $f_\sigma$ is still an isomorphism for infinitely many $\alpha \in \bar{\mathbb{Q}}$. This gives the claim.

**Theorem 4.5** (Weil) Let $X$ be defined over a finite extension $L$ of $M := M(X)$. Then $X$ can be defined over $M$ itself if and only if $\forall \sigma \in \text{Gal } \bar{M}/M$ there is an isomorphism $f_\sigma : X \to X^\sigma$ such that $\forall \sigma, \tau \in \text{Gal } \bar{M}/M$ we have
\[f_{\sigma \tau} = f^\tau \circ f_\sigma.\]
Analogous statement holds for \( M(D) \) and the field of definition for \( X \) and \( \beta \), with diagram

\[
\begin{array}{ccc}
X & \stackrel{f_\sigma}{\longrightarrow} & X^\sigma \\
\beta & \downarrow & \beta^\sigma \\
& \mathbb{P}^1(\mathbb{C}) & \\
\end{array}
\]

commuting.

Consequence: If \( \text{Aut} \ X = \{\text{id}\} \), then \( X \) is defined over \( M(X) \). \( \Leftarrow \) \( f_\sigma \) is unique (generic case for \( g > 2 \)).

**Theorem 4.6 (Coombes/Harbater, Dèbes/Emsalem, J.W., B. Köck)**

*Quasiplatonic curve \( X \) can be defined over \( M(X) \).*

**Idea:** The canonical projection \( X \to \text{Aut} \ X \setminus X \cong \mathbb{P}^1(\mathbb{C}) \) is a Belyi function, assume that the critical points are 0, 1, \( \infty \). Let \( D \) be the corresponding (regular) dessin on \( X \), \( M(D) \subset \hat{\mathbb{Q}} \). Prove first that \( X, \beta \) are defined over \( M(D) \).

Let \( r \neq 0, 1, r \in M(D) \subset \mathbb{C} \subset \hat{\mathbb{C}} \), fix one \( x \in \beta^{-1}(r), \sigma \in \text{Gal} \ \hat{\mathbb{Q}}/M(D) \) to make the following diagram commute. \( \sigma(r) = r \Rightarrow \sigma(x) \in (\beta^\sigma)^{-1}(r) \), choose \( f_\sigma \) so that \( f_\sigma(x) = \sigma(x) \Rightarrow \) unique choice for \( f_\sigma \) and it has been shown that Weil’s conditions are satisfied! The proof that \( X \) can be defined even over \( M(X) \subseteq M(D) \) needs some additional arguments.

**4.1 Problem**

7 Show that the elliptic curve can be defined over \( \mathbb{Q}(j(\tau)) \subseteq \mathbb{Q}(g_2(\tau), g_3(\tau)) \).

8 Suppose \( X \) defined over \( \overline{\mathbb{Q}} \), \( g(X) > 1 \). (\( \text{Aut} \ X \) finite \( \Rightarrow \)) Show that all automorphisms \( f : X \to X \) are also defined over \( \overline{\mathbb{Q}} \).