Lecture 11

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1 Regular Embeddings of Complete Bipartite Graphs

1.1 Regular Maps

Every bipartite map $\mathcal{B}$ is a quotient of a regular bipartite map $\widetilde{\mathcal{B}}$ by some $A \leq \operatorname{Aut} \widetilde{\mathcal{B}}$. Similarly for maps $\mathcal{M}$. Hence the importance of regular maps. One can try to classify these by type, group, genus or graph.

a) Classifying by type: Study finite quotients of the triangle group $\Delta = \Delta(l, m, n)$ of a given type (see previous lectures for some ideas). If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \leq 1$ (so Δ is infinite) then a theorem by Mal'cev states that a finitely-generated linear group is residually finite ($\bigcap\{\, K \trianglelefteq \Delta \mid |\Delta : K| < \infty \,\} = 1$), so Δ has infinitely many $K \trianglelefteq \Delta$ of finite index, so we get infinitely many regular maps of type (l, m, n).

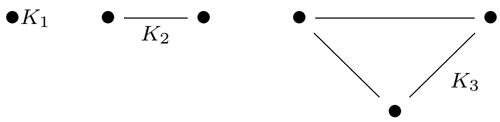
Example 1.1. Taking $\Delta = \Delta(3, 2, 7)$ we get infinitely many Hurwitz groups and Hurwitz surfaces. E.g. Macbeath (1964): $\mathrm{PSL}_2(\mathbb{F}_q)$ is a Hurwitz group $\Leftrightarrow q = 7$, or $q = p \equiv \pm 1 \mod 7$ (p prime), or $q = p^3$, prime $p \equiv \pm 2, \pm 3 \mod 7$. Conder ($\sim$ 1980): A_n is a Hurwitz group for all $n \geq 168$ (and some $n < 168$).

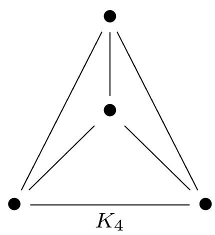
b) Classifying by group: Difficulty: G usually has many generating pairs x, y. J.D. Dixon ($\sim$ 1964): If x, y are randomly-chosen elements of $G = S_n$, then they generate either S_n or A_n with probabilities $\to \frac{3}{4}$ (not both elements even) or $\frac{1}{4}$ (both elements even) as $n \to \infty$. There are similar results for other classes of groups.

c) Classifying by genus: If $g = 0$ or 1 there are infinitely many regular maps, but they are well-known (e.g. see Chapter 8 of Coxeter and Moser for $g = 1$). If $g > 1$, Hurwitz's bound $|G| \leq 84(g - 1)$ implies that there are only finitely many regular maps of genus g, and these can by classified by hand (for small g) or computer (for larger g).

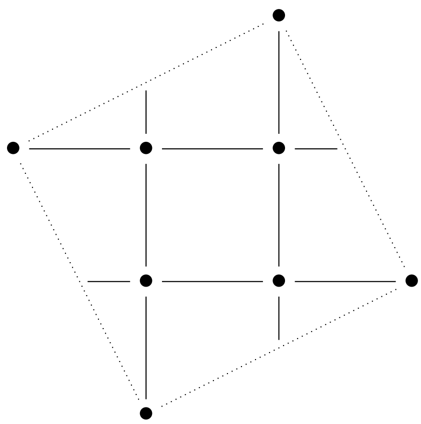
d) <u>Classifying by graph:</u> *Problem*: Given a graph $\mathcal{G}$ (or class of graphs $\mathcal{G}$), find all regular maps with $\mathcal{G}$ as the embedded graph. Equivalently, look for $G \leq \operatorname{Aut}\mathcal{G}$, transitive on the vertex-set V, with vertex-stabiliser G_v ($v \in V$) cyclic and transitive on the neighbours of v (induced by rotating the surface around v). Such an embedding can exist only if $\mathcal{G}$ is arc-transitive, i.e. $\operatorname{Aut}\mathcal{G}$ acts transitively on the arcs (=directed edges) of $\mathcal{G}$.

Example 1.2. Take $\mathcal{G} = K_n =$ "complete graph on n vertices". The graphs





are regular embeddings, $g = 0$.



$K_5 \hookrightarrow$"torus", $g = 1$. For K_6 there's no embedding and for K_7 two examples on torus (see if you can find them, imitating K_5).

Theorem 1.1 (Biggs, 1971). *K_1 has a regular embedding $\Leftrightarrow n = p^e$ for some prime p.*

Theorem 1.2 (James & J. 1985). *Regular embeddings of K_n classified and enumerated.*

Biggs's examples are the onlny ones. His construction: Take $V = \mathbb{F}_n$, finite field of order $n = p^e$, unique up to isomorphism. Multiplicative group $\mathbb{F}_n^* = \mathbb{F}_n \backslash \{0\}$ is cyclic, so choose a generator α. The cyclic order of neighbours of each vertrex v is $v+1$, $v+\alpha$, $v+\alpha^2$, ..., $v+\alpha^{n-2}$. Check that this gives a regular embedding $\mathcal{M}(\alpha)$,

$$G = \mathrm{Aut}\,\mathcal{M}(\alpha) \cong \mathrm{AGL}_1(\mathbb{F}_n) = \{\, t \mapsto at+b \mid a,b \in \mathbb{F}_n, a \neq 0 \,\}.$$

$$\mathcal{M}(\alpha) \cong \mathcal{M}(\alpha') \Leftrightarrow \alpha, \alpha' \text{ are conjugate under } \mathrm{Gal}(\mathbb{F}_n) \cong C_e,$$

where C_e is generated by the Frobenius automorphism $t \mapsto t^p$. Therefore

$$\#\text{maps } \mathcal{M}(\alpha) = \frac{\phi(n-1)}{e},$$

where $\phi(n-1)$ is the number of choices of α generating $\mathbb{F}_n^* \cong C_{n-1}$ and e is the size of orbits of $\mathrm{Gal}(\mathbb{F}_n)$.

Hint for K_7: Dessin represents the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$. Black/white vertices = 7 points & 7 lines. Can you get a regular embedding of K_4 from this? Can you get two of them?

1.2 Complete Bipartite Graphs

Take $\mathcal{G} = K_{n,n}$ ="complete bipartite graph with n black vertices and n white vertices, every black and white pair joined by one edge", so $|V| = 2n$ and $|E| = n^2$.

We look for embeddings $\mathcal{M}$ of $K_{n,n}$ which are regular as *maps*, not just as bipartite maps, i.e. $\mathrm{Aut}\,\mathcal{M}$ (ignoring the vertx-colours) should act transitively on directed edges, not just on edges, so $\mathrm{Aut}\,\mathcal{M} = \mathrm{Aut}\,\mathcal{B} \rtimes C_2$ where $\mathrm{Aut}\,\mathcal{B} = G$ is the automorphism group of the dessin (preserving vertex-colours), and C_2 reverses them.

Example 1.3. The Fermat curve $x^n + y^n = 1$, with Belyi function $\beta(x,y) = x^n$, gives a regular embedding of $K_{n,n}$ of genus $g = \frac{(n-1)(n-2)}{2}$, with $G = C_n \times C_n$ acting by sending (x,y) to $(x\zeta_n^j, y\zeta_n^k)$; here the automorphism $(x,y) \mapsto (y,x)$ transposes black and white vertices, giving $\mathrm{Aut}\,\mathcal{M} = (C_n \times C_n) \rtimes C_2$ (isomorphic to wreath product of C_n and C_2, $C_n \wr C_2$). This is the standard embedding S_n of $K_{n,n}$.

Thus if $\nu(n) :=$ "number of regular embeddings of $K_{n,n}$ (up to isomorphism). Then $\nu(n) \geq 1$ for all n, since $\mathcal{S}_n$ exists for all n.

Theorem 1.3 (Nedelar, Škoviera & J. $\sim$ 2001). $\nu(n) = 1$ *(i.e. $\mathcal{S}_n$ is the only regular embedding of $K_{n,n}$)* $\Leftrightarrow (n, \phi(n)) = 1 \Leftrightarrow n = p_1 \dots p_k$, *$p_i$ distinct primes, $p_i \nmid p_j - 1$ when $i \neq j$.*

(Compare with a result of Burnside, ~ 1900: these are the n for which there is only one group of order n, namely C_n. The proofs are independent.)

The asymptotic density of these integers n is (by Erdős, 1948)

$$\frac{\text{number of such integers } n \leq N}{N} \sim \frac{e^{-\gamma}}{\log\log\log N} \text{ as } N \to \infty$$

where γ is Euler's constant.

Theorem 1.4 (Nedela, Škoviera & J.). *If $n = p^e$, prime $p > 2$, then $\nu(n) = p^{e-1}$.*

These maps all have genus $g = \frac{(n-1)(n-2)}{2}$. They have valency n, and the faces are all $2n$-gons. The groups $G = \operatorname{Aut}\mathcal{B}$ (preserving the vertex-colours) have the form

$$G = G_f = \langle\, g, h \,|\, g^n = h^n = 1, g^{-1}hg = h^{1+p^f} \,\rangle,$$

where $f = 1, 2, \ldots, e$. (If $f = e$ then $p^f = p^e = n$, so $h^{1+p^f} = h$, so $G = C_n \times C_n$ with $\mathcal{M} = \mathcal{S}_n$)

For a given $G = G_f$, the maps $\mathcal{M}$ correspond to orbits of $\operatorname{Aut} G$ on pairs of elements x, y such that

1. $G = XY$ with $X \cap Y = 1$, $X = \langle x\rangle$ and $Y = \langle y\rangle$, both of order n;
2. some $\alpha \in \operatorname{Aut} G$ transposes x and y.

(Here x and y represent rotations around a black and a white vertex, α is conjugation of G by an automorphism of $\mathcal{M}$ reversing the edge between them.) As representatives of the orbits of $\operatorname{Aut} G$ on such pairs, one can take $x = g^u$, $y = g^u h$ (or $(gh)^u$, more convinient for Galois theory) where $u = 1, 2, \ldots, p^{e-f}$ and $(u, p) = 1$.

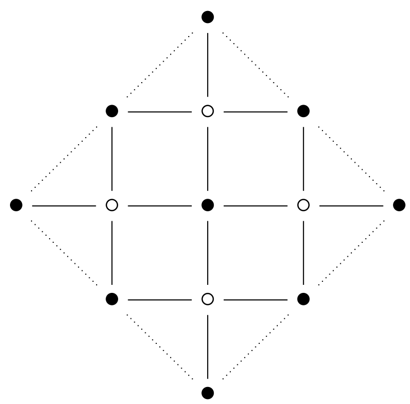
For each f we have $\phi(p^{e-f})$ possible choices of u, so summing over $f = 1, \ldots, e$ we get $\sum_{f=1}^{e} \phi(p^{e-f}) = p^{e-1}$ maps $\mathcal{M}$. These correspond to normal subgroups $K \trianglelefteq \Delta(n, n, n)$, which are also normal in $\Delta(n, 2, 2n)$ which contans $\Delta(n, n, n)$ with index 2.

$$K \trianglelefteq \Delta(n, n, n) \leq \Delta(n, 2, 2n)$$

The proof depends on:

Theorem 1.5 (Huppert, 1951). *If G is a p-group ($|G|$ is a power of p) for a prime $p > 2$, and $G = XY$ for cyclic subgroup X and Y, then G is metacyclic (i.e. there is a cyclic $N \trianglelefteq G$ with G/N cyclic).*

In our case $|G| = n^2 = p^{2e}$, and we can take $N = \langle h \rangle$. There are exceptions to Huppert's Theorem when $p = 2$, and there are also exceptional regular embeddings for $n = 2^e$. E.g. $n = q = 2^2$:



These is an embedding $\mathcal{N}_4$ of $K_{4,4}$ of genus $g = 1 \neq \frac{(n-1)(n-2)}{2}$.

Theorem 1.6 (Du, Kwak, Nedela, Škoviera & J. ~ 2005). *The regular embeddings of $K_{n,n}$ for $n = 2^e$ are:*

- *these corresponding to G_f for $f = 2, 3, \ldots, e$ (not $f = 1$).*
- *$\mathcal{N}_4$ if $e = 2$.*
- *four similar exceptions for each $e \geq 3$.*

Recent result (Apice 2006): complete classification for *all* n.

What about the associated algebraic curves, Galois orbits, fields of definition, etc. Jürgen, Manfred Streit, Antoine Coste.