Lecture 12

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5 Generalised Fermat Curves

Theorem 5.1 *Let X be a quasiplatonic curve with a regular dessin D of type (l, m, n), and suppose that for any dessin D' of the same type* $\operatorname{Aut} D' \cong \operatorname{Aut} D$ *implies $D \cong D'$. Then X can be defined over $\mathbb{Q}$.*

Proof. l, m, n is invariant under the action of $\underline{\underline{G}}_{\mathbb{C}}$ or $\operatorname{Gal} \bar{\mathbb{Q}}/\mathbb{Q}$, and moreover

$$\operatorname{Aut} X^\sigma \cong \operatorname{Aut} X.$$

$\Rightarrow \operatorname{Aut} D^\sigma \cong \operatorname{Aut} D$. From the hypothesis therefore follows $D^\sigma \cong D$. $\Rightarrow$ $\underline{\underline{G}}(D) = \underline{\underline{G}}_{\mathbb{C}} \Rightarrow M(D) = \mathbb{Q} \Rightarrow X$ can be defined over $\mathbb{Q}$. $\square$

Theorem 5.1 applies to all quasiplatonic curves up to $g = 5$.

Recall: Fermat curves F_n, $n > 3$, have a regular dessin of type (n, n, n), based on $K_{n,n}$. Suppose now that $n = p^e > 3$ (odd prime power) and suppose that all dessins are based on $K_{n,n}$, and they are regular as *maps* (i.e. there is edge-transitive automorphism group and moreover a colour-exchanging (orientable) involution $\circ \longleftrightarrow \bullet$. Recall that then [G.J., Nedela, Škoviera]

$$\operatorname{Aut} D \cong C_n \rtimes C_n := \langle\, g, h \,|\, g^n = h^n = 1,\ h^g := g^{-1}hg = h^{1+p^f} \,\rangle = G_f$$

(colour-preserving subgroup) for some $f = 1, \ldots, e$

- the G_f are pairwise non-isomorphic
- $\forall\ f = 1, \ldots, e$ there are quotients $\Delta/K_{f,u} \cong G_f$ for $\Delta = \langle n, n, n\rangle$ by the kernel $K_{f,u}$ of the homomorphism $\gamma_0 \mapsto g^u$, $\gamma_1 \mapsto (gh)^u$ for some u coprime to p
- these kernels $K_{f,u}$, $K_{f,v}$ are different $\Leftrightarrow u \not\equiv v \mod p^{n-f}$ (giving $p^{e-f} - p^{e-f-1}$ different surface groups if $f < e$)
- the case $f = e$ we have $G_e \cong C_n \times C_u \Rightarrow K_{e,1} = [\Delta, \Delta]$ and $K_{e,1}\backslash\mathbb{H} \cong F_n$.

Call $X_{f,u} := K_{f,u}\backslash\mathbb{H}$ for $f = 1, \dots, e$ and $u \in (\mathbb{Z}/p^{e-f}\mathbb{Z})^*$ "generalised Fermat curves".

Theorem 5.2 (G.J.,Manfred Streit, J.W.) *For fixed $n = p^e$ odd and $f \in \{1, \dots e\}$, these $X_{f,u}$ form one Galois orbit. Their moduli field $M(X_{f,u})$ (= a minimal field of definition) is*

$$\mathbb{Q}(\eta), \ \eta = \exp\left(\frac{2\pi i}{p^{e-f}}\right).$$

<u>Ideas for the proof:</u>

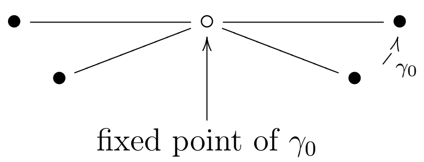
1. Show that $X_{f,u} \cong X_{g,v} \Leftrightarrow f = g$ and $u \equiv v \mod p^{e-f}$. $G_f \cong G_g \Leftrightarrow f = g$. Therefore the first implication is done. $X_{f,u} \cong X_{f,v} \Leftrightarrow K_{f,u}$ and $K_{f,v}$ conjugate in $\mathrm{PSL}_2\,\mathbb{R}$ (even in $\langle 2, 3, 2n\rangle$). That possibility can be excluded.

2. For all $\sigma \in \mathrm{Gal}\,\bar{\mathbb{Q}}/\mathbb{Q}$ consider $X_{f,u}^\sigma$.

$$\left.\begin{array}{r}\text{Ramifications are preserved}\\ \text{Regularity is preserved}\\ \mathrm{Aut}\,X_{f,u}\text{ is preserved}\end{array}\right\} \Rightarrow X_{f,u}^\sigma \cong \text{ some } X_{f,v}, \ v \in (\mathbb{Z}/p^{e-f}\mathbb{Z})^*.$$

   $\Rightarrow$ Galois orbits are parts of $\{ X_{f,u} \mid u \in (\mathbb{Z}/p^{e-f}\mathbb{Z})^* \}$

3. Acting on $X_{f,u}$ (and the dessin), g has p^f (white) fixed points and gh has p^f (black) fixed points. G_f is considered as automorphism group $\cong \Delta/K_{f,u}$, for the



$$\gamma_0 \mapsto g^u \Rightarrow g = (g^u)^{u'}$$

   (where $uu' \equiv 1 \mod n$) number of the fixed points calculate the index

$$(N_{G_f}(\langle g\rangle) : \langle g\rangle) = p^f = (N_{G_f}(\langle hg\rangle) : \langle hg\rangle)$$

4. Locally in the fixed points, γ_0 and γ_1 behave like $z \mapsto \zeta_n z+$ "higher z-powers". Fixed point $\leftrightarrow 0 = z$. $\Rightarrow g^u$ has also multiplier $\zeta = \zeta_n$ in the corresponding fixed point $\Rightarrow g$ has multiplier $\zeta^{u'}$ in the corresponding

fixed point. All g-fixed points form an orbit under $\left\langle h^{p^{e-f}} \right\rangle$-orbit and g has the same multiplier $\zeta^{u'}$ in all its fixed points! Also gh has the multiplier $\zeta^{u'}$ ($u'u \equiv 1 \mod n$) in all its • fixed points. ⇒ In its family, $X_{f,u}$ is characterised by the multipliers $\zeta^{u'}$ of g and gh in their fixed points. Fixed points of g: action locally by $z \mapsto \zeta^{u'} z+$ "higher terms".

5. Behaviour of the multipliers under $\sigma \in \operatorname{Gal} \bar{\mathbb{Q}}/\mathbb{Q}$. Suppose $g \in \operatorname{Aut} X$ with fixed points $P \in X$ and multiplier $\xi \Rightarrow$ on X^σ, g has fixed point P^σ with multiplier $\sigma(\xi)$. Choose for the local chart some $X \to \hat{\mathbb{C}}$ globally meromorphic, defined over $\bar{\mathbb{Q}}$, $P \mapsto 0$ simple zero in P, multipliers are always roots of unity (⇐ finite order), $\xi \mapsto \sigma(\xi)$ root of unity, same order.

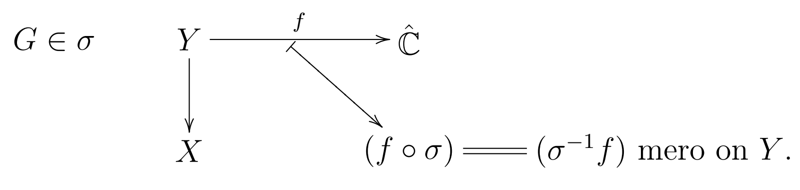
6. $\sigma \in \operatorname{Gal} \bar{\mathbb{Q}}/\mathbb{Q}$, $\sigma|_{\mathbb{Q}(\eta)} = \mathrm{id}_{\mathbb{Q}(\eta)}$, where $\eta = e^{\frac{2\pi i}{p^{e-f}}}$. $\sigma(\zeta^{u'}) = \zeta^{v'}$ (primitive n^{th}-root of unity) with $v' \equiv u' \mod p^{e-f} \Leftrightarrow v \equiv u \mod p^{e-f} \Leftrightarrow X_{f,u} \cong X_{f,v} \Rightarrow \sigma \in \underline{\underline{G}}(X_{f,u}) \Rightarrow \mathbb{Q}(\eta)$ is a field of definition. $\operatorname{Gal} \bar{\mathbb{Q}}/\mathbb{Q}$ acts transitively on the primitive n^{th} roots of unity ⇒ acts transitively on $\{X_{f,u}\}$ ⇒ they form a Galois orbit. $M(X_{f,u}) = \mathbb{Q}(\eta)$ is a field of definition.

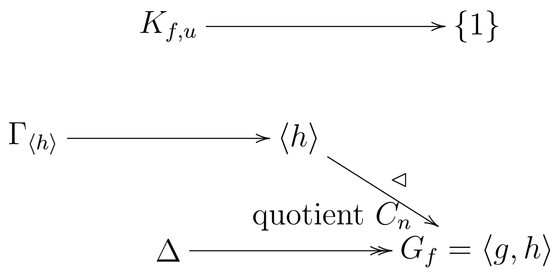
Theorem 5.3 *Let $n = p^e > 3$ be an odd prime power and $f \geq \frac{e}{2}$. Then we have a (singular, affine) model for $X_{f,1}$, given by the equations*

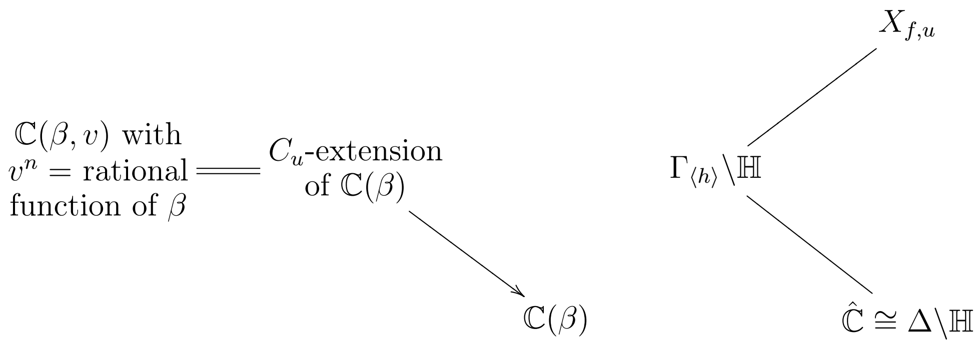
$$
\begin{aligned}
v^n &= \beta(\beta - 1) \\
w^{p^{e-f}} &= 1 - \beta \\
z^{p^f} &= w^{-r} \prod_{k=0}^{p^{e-f}-1} (w - \eta^k)^{ak}
\end{aligned}
$$

where $a := p^{2f-e}$, $r := \left((1+p^f)^{p^{e-f}} - 1\right)/p^e$ ($\in \mathbb{N}$, coprime to p).

<u>Idea:</u> Covering groups ↔ Galois groups of extensions of function fields.







Equations for $X_{f,u} = K_{f,u}\backslash\mathbb{H} \Leftarrow$ algebraic relations in the corresponding function field. This is the composite field of the function field for $\Gamma_{\langle h\rangle}\backslash\mathbb{H}$ and $\Gamma_{\langle g\rangle}\backslash\mathbb{H}$ where $\Gamma_{\langle h\rangle}$ and $\Gamma_{\langle g\rangle}$ are the preimages of $\langle h\rangle$ and $\langle g\rangle$ under the epimorphism $\Delta \to G_f$. Since

$$\Gamma_{\langle h\rangle}\backslash\mathbb{H} \to \Delta\backslash\mathbb{H} = \hat{\mathbb{C}}$$

is a cyclic covering ramified with multiplicity n above $0, 1, \infty$, with an additional symmetry between 0 and 1, we get $\mathbb{C}(\beta, v)$ as function field for the covering space with $v^n = \beta(\beta - 1)$. The corresponding construction for $\Gamma_{\langle g\rangle}\backslash\mathbb{H}$ is more complicated, since it needs a two-step tower of normal coverings.