Lecture 3

by Prof. Jürgen Wolfart

notes by Tuomas Puurtinen

1.6 Why Riemann Surfaces are Projective Algebraic Curves?

Sketch of ideas:

1. On a compact Riemann surface there are non-constant meromorphic functions $f : X \to \hat{\mathbb{C}}$. "$\Leftarrow$" by the theorem of Riemann-Roch.

2. All functions $g : X \to \hat{\mathbb{C}}$ constant on fibers of f are rational functions, in $\mathbb{C}(f)$. (Fiber means the points in $f^{-1}(q)$ for $q \in \hat{\mathbb{C}}$.)

3. Riemann-Roch: There are $h : X \to \hat{\mathbb{C}}$ separating points of the fibers of f for generic $q \in \hat{\mathbb{C}}$.

4. Consider elementary symmetric combinations

$$\begin{aligned} S_1 &= h(p_1) + h(p_2) + \ldots + h(p_n), \text{ if } \{p_i\} = f^{-1}(q) \text{ outside ram. pts of } f, \\ S_2 &= h(p_1)h(p_2) + \ldots + h(p_{n-1})h(p_n) \\ &\vdots \\ S_n &= h(p_1) \cdot \ldots \cdot h(p_n). \end{aligned} \tag{1}$$

   These are meromorphic functions on X, constant on fibers of f.

5. There exists an algebraic equation between h and f, i.e.

$$h^n - S_1 h^{n-1} + S_2 h^{n-2} - \ldots + (-1)^n S_n = 0, \tag{2}$$

   where the left side is in $\mathbb{C}(f)[h]$.

6. Algebra $\Rightarrow$ every meromorphic function on X lies in a field extension $\mathbb{C}(X)$ of $\mathbb{C}(f)$ of degree at most n.

7. Function fields determine equations $\Rightarrow$ use values of f and of h as coordinates for the curve equation 2 for the curve, resolve singularities pass to projective equation $\Rightarrow \square$.

2 Belyi functions

Theorem 2.1 *Let X be a compact Riemann surface, i.e. a smooth projective algebraic curve from $\mathbb{P}^N(\mathbb{C})$. X can be defined over $\bar{\mathbb{Q}}$, if and only if, there exists meromorphic non-constant function $\beta : X \to \hat{\mathbb{C}}$ ramified above at most 3 points (critical values are without loss of generality 0,1 and ∞).*

There functions are called "Belyi functions" (Belyi 1980)

2.1 Existence of Belyi Functions: Simple Examples

1. $\hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$, $\beta : \hat{\mathbb{C}} \to \hat{\mathbb{C}} : z \mapsto z$ (unramified).

2. $\beta : \hat{\mathbb{C}} \to \hat{\mathbb{C}} : z \mapsto z^n$ is ramified in $z = 0$ and $z = \infty$.

3. Recall Tchebychev polynomials

$$\begin{aligned} T_0(z) &\equiv 1, \\ T_1(z) &= z, \\ T_2(z) &= 2z^2 - 1 \\ &\vdots \\ T_{n+1}(z) &= 2zT_n(z) - T_{n-1}(z) \end{aligned}$$

Now $\cos n\vartheta =: T_n(\cos\vartheta)$ with properties $T_n : [-1, 1] \to [-1, 1]$, $\deg T_n = n$, T_n has simple zeros in points $\cos \frac{2k-1}{2n}\pi$, where $k = 1, \ldots, n$ and double ± 1 in between. Also simple ± 1 values at points ± 1. Now the square T_n^2 has double zeros, double 1-values in between, simple 1-values at ± 1. Therefore T_n^2 (for $n > 0$) is a Belyi function.

$$\#\text{ramification points in } \mathbb{C} \quad = \quad \#\text{zeros of } (T_n^2)' \quad = \quad 2n - 1.$$

The picture of $\beta^{-1}([0, 1])$ is $\cdots$ —— $\bullet$ —— $\circ$ —— $\cdots$ if the zeros of β are shown as $\circ$ and zeros of $\beta - 1$ as $\bullet$.

4. $X = F_n : x^n + y^n = z^n$. Then for example $\beta : F_n \to \hat{\mathbb{C}} : [x, y, z] \mapsto \frac{x^n}{z^n}$. On the affine part $z = 1$, $\beta : (x, y) \mapsto x^n$, $\deg\beta = n^2$. Less than n^2 points in $\beta^{-1}(x^n)$ occur in

   - points with $x^n = 0 \Rightarrow y = \zeta_n^2$ (n points with $\beta(x, y) = 0$, ramification order is n),
   - points with $x^n = 1 \Rightarrow y = 0$, $\beta : (x, y) = (\zeta_n^k, 0) \mapsto 1$, ram. order is n.

- $z = 0$, consider $\frac{1}{\beta} = \frac{z^n}{x^n}$: take $x = 1$, gives n zeros, all of $\operatorname{mult}_p \beta = n$.

Therefore β is a Belyi function.

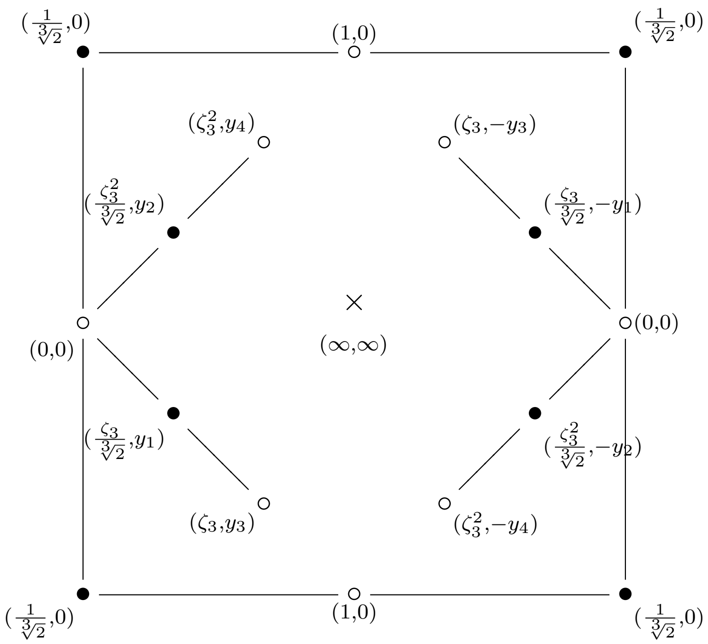
A surjective Belyi function of a compact Riemann surface $\beta : X \to \hat{\mathbb{C}}$ induces a natural triangulation by $\beta^{-1}(\hat{\mathbb{R}})$ where the preimage $\beta^{-1}(\underset{0}{\circ} \text{———} \overset{1}{\bullet})$ divides X in simply connected cells. It gives a bipartite graph embedded in X, which is called "dessin d'enfants" (by Grothendieck).

If β is a Belyi function, then also $\frac{1}{\beta}$, $1 - \beta$, $1 - \frac{1}{\beta}$, $\frac{1}{1-\beta}$, $\frac{\beta}{\beta-1}$ are Belyi functions. These permute the critical values $0, 1, \infty$.

If β is a Belyi function, then also is $4\beta(1-\beta)$: that's because $\infty \mapsto \infty$, $0 \mapsto 0$, $1 \mapsto 0$, $\frac{1}{2} \mapsto 1$. Also $\beta \mapsto 4\beta(1-\beta)$ from $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Belyi function. This last action induces a new bipartite graph, which can be reduced to simpler one-colour one. This corresponds to the theory of "maps".

2.2 Another Example

For the construction of Belyi functions $y^2 = x(x-1)(x - \frac{1}{\sqrt[3]{2}})$ elliptic function defined $/\bar{\mathbb{Q}}$. Start with $(x, y) \mapsto x$, study ramification points: this mapping is ramified in $(\infty, \infty), (0,0), (1,0), (\frac{1}{\sqrt[3]{2}}, 0)$. The preimage of the unit interval in β is of following shape:



Now

$$
\begin{aligned}
(x,y) \mapsto x \mapsto x^3 \mapsto 4x^3(1-x^3)\\
\infty \mapsto \infty\\
0 \mapsto 0\\
1 \mapsto 1\\
\frac{1}{\sqrt[3]{2}} \mapsto \frac{1}{2}.
\end{aligned}
$$

This step (using polynomials sending algebraic critical values) may induce new ramifications, here $x \mapsto x^3$ ramified in $x = 0$ and $x = \infty$, so the composite $(x,y) \mapsto x^3$ is ramifield above $0, 1, \infty, \frac{1}{2}$. The composite map will be a Belyi function sending

$$
\begin{aligned}
(\infty,\infty) &\mapsto \infty \qquad &&\text{(pole of order 12)}\\
(0,0) &\mapsto 0 &&\text{(multiplicity 6)}\\
(1,0) &\mapsto 0 &&\text{(multiplicity 2)}\\
(\zeta_3^k, \pm y_k) &\mapsto 0 &&\\
(\frac{1}{\sqrt[3]{2}}, 0) &\mapsto 1 &&\text{(multiplicity 4).}
\end{aligned}
$$

Belyi algorithm to construct β systematically: Take an equation defined $/\bar{\mathbb{Q}}$ some function $X \to \hat{\mathbb{C}}$, ramified above $\alpha_1, \ldots, \alpha_n \in \bar{\mathbb{Q}}$. Combine with a polynomial $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ sending $\alpha_1, \ldots, \alpha_n \to \mathbb{Q}$ if new ramifications arise, repeat the procedure... All critical points $\subset \mathbb{Q}$, suppose $0, 1, \infty \subset$ "crit. points". For example if $0 < \frac{m}{n+m} < 1$ is a critical point, apply $z \mapsto \frac{(m+n)^{m+n}}{m^n n^n} z^n (1-z)^n$. Then

$$
\begin{aligned}
0 \mapsto 0\\
1 \mapsto 0\\
\infty \mapsto \infty\\
\frac{m}{m+n} \mapsto 1.
\end{aligned}
$$

Only ramification occurs in $0, 1, \frac{m}{n+m}, \infty$.

2.3 Problems

2.5 (continued from 2.) Find a Belyi function and a nice dessin picture for $y^2 = x^n - 1$, $n > 3$.

3. Find a Belyi function and a dessin for the elliptic curve

$$y^2 = x(x-1)(x - \frac{\zeta_3^{\pm 1}}{\sqrt[3]{2}}).$$