Lecture 4

by Prof. Gareth Jones

notes by Tuomas Puurtinen

1.3 More on Tori

Recall the correspondence between the isomorphism classes of elliptic curves and the orbits of Γ on $\mathbb{H}$.

We would like a "nice" function on $\mathbb{H}$, taking a single value on each orbit of Γ, and different values on different orbits. We can regard g_2, g_3 and $\Delta = g_2^3 - 27g_3^2$ as functions of $\tau \in \mathbb{H}$ by evaluating them for the lattice $\Lambda = \Lambda(1, \tau)$ with $\omega_2 = \tau$ and $\omega_1 = 1$, and with modulus τ. Difficulty: if replace Λ with a similar lattice $\Lambda' = \mu\Lambda$ then g_2, g_3 are multiplied by μ^{-4} and μ^{-6}, and Δ by μ^{-12}. But if we define

$$J(\tau) = \frac{g_2(\tau)^3}{\Delta(\tau)} = \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}$$

then the powers of μ cancel, so $J(\tau)$ depends only on the similarity class of Λ. Also, g_2, g_3 and hence J are independent of the basis of Λ. So J is invariant under the action of Γ on $\mathbb{H}$, i.e.

$$J(T(\tau)) = J(\tau)$$

for all $\tau \in \mathbb{H}$ and $T \in \Gamma$. J is the *elliptic modular function* (but not an elliptic function!). J is holomorphic on $\mathbb{H}$, and it induces a bijection between the orbits of Γ on $\mathbb{H}$ and complex numbers, i.e. $\Gamma\backslash\mathbb{H} \leftrightarrow \mathbb{C}$.

Exercise 1.4 *Evaluate $J(\tau)$ at $\tau = i$ and $\tau = \omega = e^{2\pi i/3}$ and find the corresponding elliptic curves.*

1.4 Alternative Approach to Finding a "Nice" Function

Put each elliptic curve E into Legendre form

$$y^2 = x(x-1)(x-\lambda)$$

where $\lambda \in \mathbb{C}\backslash\{0, 1\}$ and regard λ as a function of the modulus τ corresponding to E. The difficulty here is that the Legendre form for E is not quite unique. This is because there are 6 ways of sending two of the three roots of $p(x)$ to 0 and 1, with the third going to λ, by an affine transformation.

For instance, if we replace x with $1-x$ (transposing the roots 0 and 1) the right-hand side of the Legendre equation becomes

$$(1-x)(-x)(1-x-\lambda) = -x(x-1)(x-(1-\lambda)).$$

If we also replace y with iy the left-hand side becomes $-y^2$, so we have an isomorphic elliptic curve with Legendre form

$$y^2 = x(x-1)(x-(1-\lambda)).$$

Thus λ is replaced with $1-\lambda$. Another substitution (find it!) replaces λ with $\frac{1}{\lambda}$. These two substitutions generate a group isomorphic to S_3 (corresponding to permuting the three roots e_1, e_2 and e_3 of $p(x)$), and the six permutations give rise to six values

$$\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}.$$

One can define λ uniquely as a function of τ by noting that $\wp'(z) = 0$ at $z = \frac{\omega_1}{2}$ and $\frac{\omega_1+\omega_2}{2}$ (why?), so the differential equation

$$(\wp')^2 = p(\wp)$$

implies that the roots e_1, e_2 and e_3 of $p(x)$ are at $x = \wp(\frac{\omega_1}{2})$, $\wp(\frac{\omega_2}{2})$ and $\wp(\frac{\omega_1+\omega_2}{2})$.

An affine transformation $L : x \mapsto ax+b$ sending e_2 and e_3 to 0 and 1 respectively sends e_1 to

$$\lambda = \frac{e_1 - e_2}{e_3 - e_2}$$

and this depends only on τ. This function λ is holomorphic on $\mathbb{H}$, and is invariant under $\Gamma(2)$ (a normal subgroup of index 6 in Γ), but not under Γ. The 6 cosets of $\Gamma(2)$ in Γ give the 6 possible values for λ. These two functions are related by:

$$J(\tau) = \frac{4(1-\lambda(\tau)+\lambda(\tau)^2)^3}{27\lambda(\tau)^2(1-\lambda(\tau))^2}$$

Thus six values of λ correspond to each value of J. Then

$$\beta(x) = \frac{4(1-x+x^2)^3}{27x^2(1-x)^2}$$

is a Belyi function. It has triple zeros of β at $e^{\pm 2\pi i/6} (= \zeta_6^{\pm 1})$, double zeros of $\beta - 1$ at $-1, \frac{1}{2}, 2$, and double poles of β at $0, 1, \infty$.

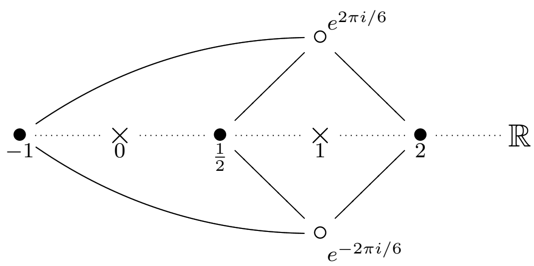
2 Embeddings of Graphs, Maps and Hypermaps

Graph $\mathcal{G} = (V, E)$ (vertices and edges), connected, finite (relax this later), allow loops and multiple edges. Map $\mathcal{M} : \mathcal{G} \hookrightarrow X$, X is a surface, connected, compact, without boundary, and oriented (chosen orientation counter-clockwise). The faces (connected components of X $\mathcal{G}$) must be simply-connected, i.e. homeomorphic to an open disc. Examples: Platonic solids on $X = S^2$.

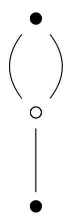
Assume that $\mathcal{G}$ is bipartite, i.e. we can colour the vertices black and white so that each edge joins a black vertex to a white vertex $\circ$ —— $\bullet$ (possible iff each circuit in $\mathcal{G}$ has even length). Call these *bipartite maps* (=dessins d'enfants) denoted by $\mathcal{B}$.

2.1 Examples of Bipartite Maps

1. The dessin $\mathcal{B}_1$ corresponding to β is

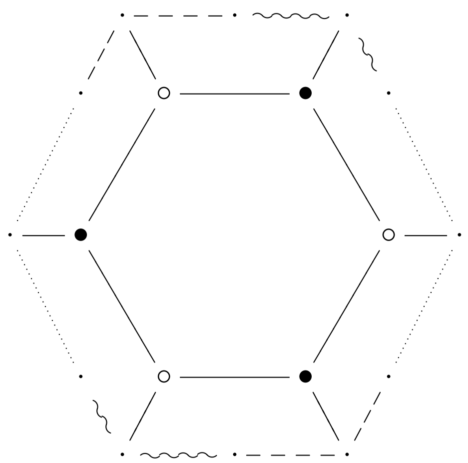


Here $\times$ denotes a face-centre.

2. $\mathcal{B}_2 =$  Quotient of $\mathcal{B}_1$ by a half-turn about $\frac{1}{2}$.

3. Identify opposite edges of the hexagon to get a bipartite map $\mathcal{B}_3$ on a

torus.



Each black and white pair are joined by a single edge, so $\mathcal{G} = K_{3,3}$, the complete bipartite graph with 3 black and 3 white vertices.

Describe $\mathcal{B}$ algebraically: use the orientation of X to define two permutations x and y of the set E of edges. For each $e \in E$, ex and ey are the next edges around the incident black and white vertices, following the orientation of X. Warning: these are not generally automorphisms.

Black vertices $\longleftrightarrow$ cycles of x on E

White vertices $\longleftrightarrow$ cycles of y on E

Faces $\longleftrightarrow$ cycles of xy on E

The orders l, m, n of x, y, xy are the least common multiples of their cycle lengths. Call (l, m, n) *the type of* $\mathcal{B}$. E.g. $\mathcal{B}_1$ and $\mathcal{B}_2$ have type $(3, 2, 2)$, $\mathcal{B}_3$ has type $(3, 3, 3)$. The *monodromy group* of $\mathcal{B}$ is the subgroup $G = \langle x, y\rangle$ generated by x and y in the symmetric group $\mathrm{Sym}(E)$ of all permutations of E.

$\mathcal{G}$ is connected, so G acts transitively on E, so the action is equivalent to the action on the cosets Hg ($g \in G$) of a stabilizer $H = G_e$ ($e \in E$). Say G acts *regularly* if $G_e = 1$; this action is equivalent to G acting on itself by right multiplication.

In $\mathcal{B}_1$, $x^3 = y^2 = (xy)^2 = 1$, and these relations define the dihedral group D_3 of order 6, so G is a quotient of D_3. G is transitive on the 6 edges, so $|G : G_e| = 6$ (the index of the subgroup), so $G \cong D_3$ with $G_e = 1$. G acts regularly. In $\mathcal{B}_2$, $G \cong D_3$, but $|G_e| = 2$, so the action is not regular.

In $\mathcal{B}_3$, $G \cong C_3 \times C_3$ acting regularly. Here $x^3 = y^3 = 1$ and $xy = yx$.

2.2 More Definitions

Algebraic bipartite map: (G, x, y, E) where $G = \langle x, y \rangle$ is a permutation group acting transitively on a set E. Reconstruct a bipartite map $\mathcal{B}$ from (G, x, y, E):

$$\begin{aligned} \text{edges} &= \text{elements of } E \\ \text{black/white vertices} &= \text{cycles of } x \text{ and } y \\ \text{faces} &= \text{cycles of } xy \end{aligned}$$

Incidence =containment in a cycle.

Exercise 2.1 *Take $x = (1, 2, \ldots, N)$ and $y = (1, 2)$ in S_N. Find $\mathcal{B}$ and G.*

An *automorphism* of $\mathcal{B}$ is a permutation of E commuting with x and y, or equivalently commuting with G. E.g. rotations for the example dessins $\mathcal{B}_1$ and $\mathcal{B}_3$, translations for $\mathcal{B}_3$, but only the identity for $\mathcal{B}_2$. The automorphisms form a group

$$\operatorname{Aut} \mathcal{B} = C(G) = C = \{\, c \in \operatorname{Sym}(E) \mid cg = gc \text{ for all } g \in G \,\},$$

the centralizer of G in $\operatorname{Sym}(E)$.

A permutation group is *semiregular* (acts freely) if each stabiliser is trivial.

$$\text{The group is } \left\{ \begin{array}{c} \text{semiregular} \\ \text{transitive} \\ \text{regular} \end{array} \right\} \text{ as } \left\{ \begin{array}{c} \text{at most} \\ \text{at least} \\ \text{exactly} \end{array} \right\} \begin{array}{c} \text{one group element} \\ \text{takes one point to another.} \end{array}$$

Thus regular $\Leftrightarrow$ transitive and semiregular.

Theorem 2.1 *Let G be any transitive group, and $C = C(G)$ its centraliser.*

(i) C acts semiregularly.

(ii) C acts regularly iff G does.

(iii) If C and G act regularly then $C \cong G$.

Proof.

(i) Let $c \in C$ fix e. Any e' has the form $e' = eg$ for some $g \in G$ by transtivity. Then $e'c = egc = ecg = eg = e'$, so $c = 1$.

(ii) Let C act regularly. Then C is transitive, so its centraliser is semiregular by (i) applied to C; but G commutes with C, so G is semiregular, and being transitive it must be regular.

Conversely, let G act regularly, so it is acting on itself by right-multiplication $\rho_g : e \mapsto eg$; then left-multiplication $\lambda_c : e \mapsto c^{-1}e$ commutes with right-multiplication ($c^{-1}(eg) = (c^{-1}e)g$), and acts transitively, so C is transitive, and C is semiregular by (i), so C is regular.

(iii) When C and G act regularly, then $\lambda_g \leftrightarrow \rho_g$ gives the isomorphism $C \cong G$.

□

A dessin $\mathcal{B}$ is *regular* if G (equivalently $\mathrm{Aut}\,\mathcal{B}$) is regular in E. From the last examples $\mathcal{B}_1$ and $\mathcal{B}_3$ are regular, $\mathcal{B}_2$ is not.

Exercise 2.2 *Show that $C \cong N_G(G_e)/G_e$ where $N_G(G_e)$ is the normaliser of G_e in G.*

Exercise 2.3 *If $\mathcal{B}$ is a regular dessin of type (l, m, n) with N edges, what is its genus? Are there finitely or infinitely many dessins of a given type and genus?*