1.3 More on Tori

Recall the correspondence between the isomorphism classes of elliptic curves and the orbits of \( \Gamma \) on \( \mathbb{H} \).

We would like a “nice” function on \( \mathbb{H} \), taking a single value on each orbit of \( \Gamma \), and different values on different orbits. We can regard \( g_2 \), \( g_3 \) and \( \Delta = g_3^2 - 27g_2^3 \) as functions of \( \tau \in \mathbb{H} \) by evaluating them for the lattice \( \Lambda = \Lambda(1, \tau) \) with \( \omega_2 = \tau \) and \( \omega_1 = 1 \), and with modulus \( \tau \). Difficulty: if replace \( \Lambda \) with a similar lattice \( \Lambda' = \mu \Lambda \) then \( g_2, g_3 \) are multiplied by \( \mu^{-4} \) and \( \mu^{-6} \), and \( \Delta \) by \( \mu^{-12} \). But if we define

\[
J(\tau) = \frac{g_2(\tau)^3}{\Delta(\tau)} = \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}
\]

then the powers of \( \mu \) cancel, so \( J(\tau) \) depends only on the similarity class of \( \Lambda \). Also, \( g_2, g_3 \) and hence \( J \) are independent of the basis of \( \Lambda \). So \( J \) is invariant under the action of \( \Gamma \) on \( \mathbb{H} \), i.e.

\[
J(T(\tau)) = J(\tau)
\]

for all \( \tau \in \mathbb{H} \) and \( T \in \Gamma \). \( J \) is the elliptic modular function (but not an elliptic function!). \( J \) is holomorphic on \( \mathbb{H} \), and it induces a bijection between the orbits of \( \Gamma \) on \( \mathbb{H} \) and complex numbers, i.e. \( \Gamma \backslash \mathbb{H} \leftrightarrow \mathbb{C} \).

**Exercise 1.4** Evaluate \( J(\tau) \) at \( \tau = i \) and \( \tau = \omega = e^{2\pi i/3} \) and find the corresponding elliptic curves.

1.4 Alternative Approach to Finding a ”Nice” Function

Put each elliptic curve \( E \) into Legendre form

\[
y^2 = x(x - 1)(x - \lambda)
\]

where \( \lambda \in \mathbb{C} \backslash \{0, 1\} \) and regard \( \lambda \) as a function of the modulus \( \tau \) corresponding to \( E \). The difficulty here is that the Legendre form for \( E \) is not quite unique. This is because there are 6 ways of sending two of the three roots of \( p(x) \) to 0 and 1, with the third going to \( \lambda \), by an affine transformation.
For instance, if we replace $x$ with $1 - x$ (transposing the roots 0 and 1) the right-hand side of the Legendre equation becomes

$$(1 - x)(-x)(1 - x - \lambda) = -x(x - 1)(x - (1 - \lambda)).$$

If we also replace $y$ with $iy$ the left-hand side becomes $-y^2$, so we have an isomorphic elliptic curve with Legendre form

$$y^2 = x(x - 1)(x - (1 - \lambda)).$$

Thus $\lambda$ is replaced with $1 - \lambda$. Another substitution (find it!) replaces $\lambda$ with $\frac{1}{\lambda}$. These two substitutions generate a group isomorphic to $S_3$ (corresponding to permuting the three roots $e_1, e_2$ and $e_3$ of $p(x)$), and the six permutations give rise to six values

$$\lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda}{1 - \lambda}, \frac{\lambda - 1}{\lambda}.$$

One can define $\lambda$ uniquely as a function of $\tau$ by noting that $\wp'(z) = 0$ at $z = \frac{\omega_1}{2}$ and $\frac{\omega_1 + \omega_2}{2}$ (why?), so the differential equation

$$(\wp')^2 = p(\wp)$$

implies that the roots $e_1, e_2$ and $e_3$ of $p(x)$ are at $x = \wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2})$ and $\wp(\frac{\omega_1 + \omega_2}{2})$.

An affine transformation $L : x \mapsto ax + b$ sending $e_2$ and $e_3$ to 0 and 1 respectively sends $e_1$ to

$$\lambda = \frac{e_1 - e_2}{e_3 - e_2}$$

and this depends only on $\tau$. This function $\lambda$ is holomorphic on $\mathbb{H}$, and is invariant under $\Gamma(2)$ (a normal subgroup of index 6 in $\Gamma$), but not under $\Gamma$. The 6 cosets of $\Gamma(2)$ in $\Gamma$ give the 6 possible values for $\lambda$. These two functions are related by:

$$J(\tau) = \frac{4(1 - \lambda(\tau) + \lambda(\tau)^2)^3}{27\lambda(\tau)^2(1 - \lambda(\tau))^2}$$

Thus six values of $\lambda$ correspond to each value of $J$. Then

$$\beta(x) = \frac{4(1 - x + x^2)^3}{27x^2(1 - x)^2}$$

is a Belyi function. It has triple zeros of $\beta$ at $e^{\pm 2\pi i/6}(= \zeta_6^{\pm 1})$, double zeros of $\beta - 1$ at $-1, \frac{1}{2}, 2$, and double poles of $\beta$ at 0, 1, $\infty$. 

2
2 Embeddings of Graphs, Maps and Hyper-maps

Graph $G = (V, E)$ (vertices and edges), connected, finite (relax this later), allow loops $\bigcirc \bullet$ and multiple edges $\bullet \bigcirc \bullet$. Map $M : G \hookrightarrow X$, $X$ is a surface, connected, compact, without boundary, and oriented (chosen orientation counter-clockwise). The faces (connected components of $X \ G$) must be simply-connected, i.e. homeomorphic to an open disc. Examples: Platonic solids on $X = S^2$.

Assume that $G$ is bipartite, i.e. we can colour the vertices black and white so that each edge joins a black vertex to a white vertex $\circ \longrightarrow \bullet$ (possible iff each circuit in $G$ has even length). Call these bipartite maps (=dessins d’enfants) denoted by $B$.

2.1 Examples of Bipartite Maps

1. The dessin $B_1$ corresponding to $\beta$ is

2. $B_2 = \begin{pmatrix} \bullet \\ \circ \end{pmatrix}$ Quotient of $B_1$ by a half-turn about $\frac{1}{2}$.

3. Identify opposite edges of the hexagon to get a bipartite map $B_3$ on a
Each black and white pair are joined by a single edge, so $G = K_{3,3}$, the complete bipartite graph with 3 black and 3 white vertices.

Describe $B$ algebraically: use the orientation of $X$ to define two permutations $x$ and $y$ of the set $E$ of edges. For each $e \in E$, $ex$ and $ey$ are the next edges around the incident black and white vertices, following the orientation of $X$. Warning: these are not generally automorphisms.

The orders $l, m, n$ of $x, y, xy$ are the least common multiples of their cycle lengths. Call $(l, m, n)$ the type of $B$. E.g. $B_1$ and $B_2$ have type $(3, 2, 2)$, $B_3$ has type $(3, 3, 3)$. The monodromy group of $B$ is the subgroup $G = \langle x, y \rangle$ generated by $x$ and $y$ in the symmetric group $\text{Sym}(E)$ of all permutations of $E$.

$G$ is connected, so $G$ acts transitively on $E$, so the action is equivalent to the action on the cosets $Hg \ (g \in G)$ of a stabilizer $H = G_e \ (e \in E)$. Say $G$ acts regularly if $G_e = 1$; this action is equivalent to $G$ acting on itself by right multiplication.

In $B_1$, $x^3 = y^2 = (xy)^2 = 1$, and these relations define the dihedral group $D_3$ of order 6, so $G$ is a quotient of $D_3$. $G$ is transitive on the 6 edges, so $|G : G_e| = 6$ (the index of the subgroup), so $G \cong D_3$ with $G_e = 1$. $G$ acts regularly. In $B_2$, $G \cong D_3$, but $|G_e| = 2$, so the action is not regular.
In $B_3$, $G \cong C_3 \times C_3$ acting regularly. Here $x^3 = y^3 = 1$ and $xy = yx$.

### 2.2 More Definitions

Algebraic bipartite map: $(G, x, y, E)$ where $G = \langle x, y \rangle$ is a permutation group acting transitively on a set $E$. Reconstruct a bipartite map $B$ from $(G, x, y, E)$:

- **edges** = elements of $E$
- **black/white vertices** = cycles of $x$ and $y$
- **faces** = cycles of $xy$

**Exercise 2.1** Take $x = (1, 2, \ldots, N)$ and $y = (1, 2)$ in $S_N$. Find $B$ and $G$.

An **automorphism** of $B$ is a permutation of $E$ commuting with $x$ and $y$, or equivalently commuting with $G$. E.g. rotations for the example dessins $B_1$ and $B_3$, translations for $B_3$, but only the identity for $B_2$. The automorphisms form a group

$$\text{Aut } B = C(G) = C = \{ c \in \text{Sym}(E) \mid cg = gc \text{ for all } g \in G \},$$

the centralizer of $G$ in $\text{Sym}(E)$.

A permutation group is **semiregular** (acts freely) if each stabiliser is trivial.

The group is

$$\begin{cases} \text{semiregular} \\ \text{transitive} \\ \text{regular} \end{cases} \text{ as } \begin{cases} \text{at most} \\ \text{at least} \\ \text{exactly} \end{cases} \text{ one group element takes one point to another.}$$

Thus regular $\iff$ transitive and semiregular.

**Theorem 2.1** Let $G$ be any transitive group, and $C = C(G)$ its centraliser.

(i) $C$ acts semiregularly.

(ii) $C$ acts regularly iff $G$ does.

(iii) If $C$ and $G$ act regularly then $C \cong G$.

**Proof.**

(i) Let $c \in C$ fix $e$. Any $e'$ has the form $e' = eg$ for some $g \in G$ by transitivity. Then $e'c = egc = ecg = eg = e'$, so $c = 1$. 

5
(ii) Let $C$ act regularly. Then $C$ is transitive, so its centraliser is semiregular by (i) applied to $C$; but $G$ commutes with $C$, so $G$ is semiregular, and being transitive it must be regular. Conversely, let $G$ act regularly, so it is acting on itself by right-multiplication $\rho_g : e \mapsto eg$; then left-multiplication $\lambda_c : e \mapsto c^{-1}e$ commutes with right-multiplication $(c^{-1}(eg) = (c^{-1}e)g)$, and acts transitively, so $C$ is transitive, and $C$ is semiregular by (i), so $C$ is regular.

(iii) When $C$ and $G$ act regularly, then $\lambda_g \leftrightarrow \rho_g$ gives the isomorphism $C \cong G$.

A dessin $B$ is regular if $G$ (equivalently $\text{Aut } B$) is regular in $E$. From the last examples $B_1$ and $B_3$ are regular, $B_2$ is not.

**Exercise 2.2** Show that $C \cong N_G(G_e)/G_e$ where $N_G(G_e)$ is the normaliser of $G_e$ in $G$.

**Exercise 2.3** If $B$ is a regular dessin of type $(l, m, n)$ with $N$ edges, what is its genus? Are there finitely or infinitely many dessins of a given type and genus?