Lecture 5

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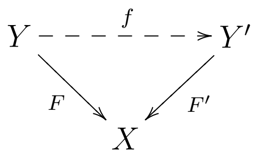
2.4 Problems

4 How does the dessin change if the Belyi function β is replaced by $1-\beta$, $\frac{1}{\beta}$? How are the pole orders encoded in the dessin? Start by some dessin, how can you modify β such that every edge $\bullet$ —— $\circ$ is replaced by $\bullet$ —— $\circ$ —— $\bullet$ —— $\circ$?

3 Uniformisation and Fuchsian Groups

3.1 Uniformisation

Theorem 3.1 *Let X be a connected manifold. There is always a "universal simply connected covering" $F : Y \to X$, where Y is a simply connected manifold with the following uniqueness property. Let F' be any other covering $Y' \to X$ and $p \in X, q \in Y$, $q' \in Y'$ s.t. $F(q) = p = F'(q')$ then there is a unique covering map $f : Y \to Y'$, such that $f(q) = q'$ making the diagram*



commute i.e. $F = F' \circ f$.

"Covering" means: $\forall p \in X \exists U = U(p)$ s.t. $F^{-1}U = \bigcup V_U$ where $F|_{V_U} : V_U \to U$ is a homeomorphism.

Consequence 3.2 *X is a Riemann surface $\Rightarrow$ Y is simply connected Riemann surface, F holomorphic and unramified.*

Contruction of Y and F is given by homotopy theory. As a set,

$$Y = \{\, (p, [\gamma]) \,|\, p \in X, [\gamma] \in \pi_1(X, p) \,\}$$

(closed curves modulo homotopy, with starting point p). Define a topology, define F as projection, control properties, in particular simple connectedness.

Theorem 3.3 (Extended Riemann Mapping Theorem, Main Theorem of Uniformisation) *If Y is simply connected, Y is isomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$ or to $\mathbb{H} \cong \mathbb{D}$ (open unit disc).*

Theorem 3.4 *Every Riemann surface X is homeomorphic to some quotient space $G\backslash Y$ where Y is the universal covering space and G is the "covering group" $\subset \operatorname{Aut} Y$ consisting of all $\gamma \in \operatorname{Aut} Y$ with $F \circ \gamma = F$ permuting transitively the fibers of F.*

($\Leftarrow$ uniqueness part of theorem 3.1.) G acts without fixed points, it is torsion free, it acts discontinuously.

Here "discontinuously" (properly) means: $\forall q \in Y \exists V = V(q)$ s.t. $V \cap \gamma V = \emptyset$ except for finitely many $\gamma \in G$.

Consequence 3.5 *a)* $Y = \hat{\mathbb{C}}$, $\operatorname{Aut} \hat{\mathbb{C}} \cong \mathrm{PSL}_2(\mathbb{C})$

$$X = \hat{\mathbb{C}} \Leftarrow G = \{\mathrm{id}\} \Leftarrow \begin{cases} z \mapsto \frac{az+b}{cz+a} \\ \textit{have fixed points} \end{cases}$$

b) $Y = \hat{\mathbb{C}}$, $\operatorname{Aut} \mathbb{C} = \{\, z \mapsto az + b \,|\, a \in \mathbb{C}^*, b \in \mathbb{C} \,\}$

$$G \subset \{\, \textit{translations} \,\} \Leftarrow \textit{no fixed point, iff } a = 1$$

It follows that G either $\cong \mathbb{Z}$ or a lattice Λ. Also $X = \mathbb{C}$ or $\mathbb{Z}\backslash\mathbb{C}$ or $\Lambda\backslash\mathbb{C}$, a torus (elliptic curve). For example in the case $X = \mathbb{Z}\backslash\mathbb{C}$

$$F(z) = \exp z, F : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} - \{0\}$$

and $\exp(z + 2\pi i k) = \exp(z)$ *for all $k \in \mathbb{Z}$.*

c) $Y = \mathbb{H} \cong \mathbb{D}$ *in all other cases, in particular for all compact Riemann surfaces X with $g > 1$. Here $G \subset \operatorname{Aut} \mathbb{H}$ and it's called "Fuchsian group".*

In general, discontinuous groups may have fixed points, i.e. points $p \in Y$ with a finite

$$G_p := \{\, \gamma \in G \,|\, \gamma(p) = p \neq \{\mathrm{id}\} \,\}.$$

Theorem 3.6 *For a discontinuous group G acting on $Y = \hat{\mathbb{C}}, \mathbb{C}, \mathbb{H}$ the following holds:*

a) for $p \in G, \gamma(p) = p$ there exist local chart such that diagram

$$\begin{array}{ccc} U(p) & \xrightarrow{\gamma} & U(p) \\ \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle z} \\ \mathbb{D} & \xrightarrow[z \mapsto \zeta_n^k z]{} & \mathbb{D} \end{array}$$

commutes and $z \mapsto z^n : \mathbb{D} \to \mathbb{D}$ induces the quotient map $U \to G_p \backslash U$ for $G_p = \langle \gamma \rangle$.

b) all stabilizing subgroups are finite cyclic

c) fixed points of G form a discrete subset

d) $G \backslash Y$ has a holomorphic structure as a Riemann surface s.t. the quotient map $Y \to G \backslash Y : z \mapsto Gz$ is holomorphic, ramified of multiplicity n in fixed points of order n.

For now $Y = \mathbb{H}$ and G discontinuous $\subset \operatorname{Aut} \mathbb{H}$.

Theorem 3.7 $\operatorname{Aut} \mathbb{H} = \mathrm{PSL}_2(\mathbb{R})$*, the group of orientation preserving hyperbolic motions.*

It is clear that $\operatorname{Aut} \mathbb{H} \supseteq \mathrm{PSL}_2 \mathbb{R}$, acting (simply) transitively on $\{$points$\}$ and $\{$lines$\}$, suppose $\gamma \in \operatorname{Aut} \mathbb{H}$ by combination with some $\mu \in \mathrm{PSL}_2 \mathbb{R}$, suppose $\gamma(i) = i$, suppose $\gamma \in \operatorname{Aut} \mathbb{D}$, and $\gamma(0) = 0$. Lemma (Schwarz): If holomorphic $\delta : \mathbb{D} \to \mathbb{D}$ has $\delta(0) = 0$, then

$$|\delta(z)| \leq |z| \ \forall z \in \mathbb{D} \text{ with "=" iff}$$

$$\delta(z) = \lambda z \text{ with } |\lambda| = 1.$$

Hence, δ and its inverse mapping satisfy both $|\delta^{\pm 1}(z)| \leq |z|$, therefore $\delta(z) = \lambda z$ is a hyperbolic motion $\Rightarrow \square$.

Theorem 3.8 $G \subset \mathrm{PSL}_2(\mathbb{R})$ *acts discontinuously on $\mathbb{H}$, iff G is a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{R})$.

3.2 Fuchsian Groups

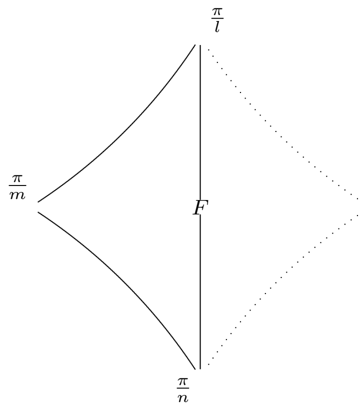
There are two methods for the contruction of Fuchsian groups:

- Arithmetic: Construct discrete groups of $\mathrm{PSL}_2 \mathbb{R}$ by numbertheory, e.g. $\Gamma = \mathrm{PSL}_2 \mathbb{Z}$ (modular group).

- Geometry (Poincaré): Start with a "suitable" hyperbolic polygon F (later serving as the fundamental domain for G), and generate G by side-pairing transformations.

Example: the "triangle groups" $\langle l, m, n\rangle$ (drawing in $\mathbb{D}$ instead of $\mathbb{H}$), $l, m, n \in \mathbb{N}\backslash\{0\}$ or ∞ with

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1.$$



For example $\langle 2, 3, \infty\rangle = \mathrm{PSL}_2\,\mathbb{Z}$ and

$$\langle \infty, \infty, \infty\rangle = \Gamma(2) = \{\, \gamma \in \Gamma = \mathrm{PSL}_2\,\mathbb{Z} \mid \gamma \equiv E \mod 2\,\}$$

Also $\Gamma/\Gamma(2) = \mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3$. Fact: there are 85 triangle groups which are "arithmetically defined" (Takeuchi $\sim$ 1970).

$$\gamma_0^l = 1 = \gamma_1^m = \gamma_\infty^n = \gamma_\infty\gamma_1\gamma_0.$$

Theorem 3.9 *a) These triangle groups $\langle l, m, n\rangle = \langle \gamma_0, \gamma_1, \gamma_\infty\rangle$ are discontinuous on $\mathbb{H}$ with F as "fundamental region" (i.e. F open and $F \cap \gamma F = \emptyset \forall \gamma \in G - \{1\}$ and $\bigcup_{\gamma\in G} \gamma\bar{F} = \mathbb{H}$) (hard work!)*

$\vdots$

d) There is a meromorphic G-invariant function $j : \mathbb{H} \to \hat{\mathbb{C}}$ ($j(\gamma(z)) = j(z) \forall z \in \mathbb{H}$ and $\gamma \in G$) mapping the two parts of F biholomorphicaly onto $\mathbb{H}$ and $-\mathbb{H}$, border edges onto $\hat{\mathbb{R}}$ and the vertices onto $0, 1, \infty$, with multiplicities l, m, n.

3.3 Problem

5 Let G be a (possibly ramified) covering group of X acting discontinuously on Y with $X \cong G\backslash Y$ and let $N \triangleleft G$ normal subgroup, $X' := N\backslash Y$. Show that G/N acts as group of automorphisms on X' s.t.

$$(G/N)\backslash X' \cong X.$$

Give an example of a Riemann surface with an automorphism group $\mathrm{PSL}_2(\mathbb{Z}/NZ)$, $N \in \mathbb{N}\backslash\{0\}$.