2.4 Problems

4 How does the dessin change if the Belyi function $\beta$ is replaced by $1 - \beta$? How are the pole orders encoded in the dessin? Start by some dessin, how can you modify $\beta$ such that every edge $\bullet \circ \bullet \circ$ is replaced by $\bullet \circ \circ \bullet$?

3 Uniformisation and Fuchsian Groups

3.1 Uniformisation

**Theorem 3.1** Let $X$ be a connected manifold. There is always a "universal simply connected covering" $F : Y \to X$, where $Y$ is a simply connected manifold with the following uniqueness property. Let $F'$ be any other covering $Y' \to X$ and $p \in X, q \in Y, q' \in Y'$ s.t. $F(q) = p = F'(q')$ then there is a unique covering map $f : Y \to Y'$, such that $f(q) = q'$ making the diagram

$$Y \xrightarrow{f} Y'$$

commute i.e. $F = F' \circ f$.

"Covering" means: $\forall p \in X \exists U = U(p)$ s.t. $F^{-1} U = \bigcup V_U$ where $F\big|_{V_U} : V_U \to U$ is a homeomorphism.

**Consequence 3.2** $X$ is a Riemann surface $\Rightarrow Y$ is simply connected Riemann surface, $F$ holomorphic and unramified.

Construction of $Y$ and $F$ is given by homotopy theory. As a set,

$$Y = \{ (p, [\gamma]) \mid p \in X, [\gamma] \in \pi_1(X, p) \}$$

(closed curves modulo homotopy, with starting point $p$). Define a topology, define $F$ as projection, control properties, in particular simple connectedness.
Theorem 3.3 (Extended Riemann Mapping Theorem, Main Theorem of Uniformisation) If $Y$ is simply connected, $Y$ is isomorphic to $\hat{\mathbb{C}}, \mathbb{C}$ or to $\mathbb{H} \cong \mathbb{D}$ (open unit disc).

Theorem 3.4 Every Riemann surface $X$ is homeomorphic to some quotient space $G \backslash Y$ where $Y$ is the universal covering space and $G$ is the "covering group" $\subset \text{Aut } Y$ consisting of all $\gamma \in \text{Aut } Y$ with $F \circ \gamma = F$ permuting transitively the fibers of $F$.

$(\Leftarrow$ uniqueness part of theorem 3.1.) $G$ acts without fixed points, it is torsion free, it acts discontinuously.

Here "discontinuously" (properly) means: $\forall q \in Y \exists V = V(q)$ s.t. $V \cap \gamma V = \emptyset$ except for finitely many $\gamma \in G$.

Consequence 3.5 a) $Y = \hat{\mathbb{C}}$, $\text{Aut } \hat{\mathbb{C}} \cong \text{PSL}_2(\mathbb{C})$

$$X = \hat{\mathbb{C}} \iff G = \{ \text{id} \} \iff \left\{ \begin{array}{l} z \mapsto \frac{az+b}{cz+a} \\
\text{have fixed points} \end{array} \right.$$  

b) $Y = \hat{\mathbb{C}}$, $\text{Aut } \mathbb{C} = \{ z \mapsto az + b | \ a \in \mathbb{C}^*, b \in \mathbb{C} \}$

$G \subset \{ \text{translations} \} \iff \text{no fixed point, iff } a = 1$

It follows that $G$ either $\cong \mathbb{Z}$ or a lattice $\Lambda$. Also $X = \mathbb{C}$ or $\mathbb{Z} \backslash \mathbb{C}$ or $\Lambda \backslash \mathbb{C}$, a torus (elliptic curve). For example in the case $X = \mathbb{Z} \backslash \mathbb{C}$

$$F(z) = \exp z, F : \mathbb{C} \to \mathbb{C}^\ast = \mathbb{C} - \{0\}$$

and $\exp(z + 2\pi ik) = \exp(z)$ for all $k \in \mathbb{Z}$.

c) $Y = \mathbb{H} \cong \mathbb{D}$ in all other cases, in particular for all compact Riemann surfaces $X$ with $g > 1$. Here $G \subset \text{Aut } \mathbb{H}$ and it’s called “Fuchsian group”.

In general, discontinuous groups may have fixed points, i.e. points $p \in Y$ with a finite

$$G_p := \{ \gamma \in G | \gamma(p) = p \neq \{\text{id}\} \}.$$  

Theorem 3.6 For a discontinuous group $G$ acting on $Y = \hat{\mathbb{C}}, \mathbb{C}, \mathbb{H}$ the following holds:
a) for $p \in G$, $\gamma(p) = p$ there exist local chart such that diagram

\[
\begin{array}{ccc}
U(p) & \xrightarrow{\gamma} & U(p) \\
\downarrow & & \downarrow \\
D & \xrightarrow{z \rightarrow c_nz} & D
\end{array}
\]

commutes and $z \mapsto z^n : \mathbb{D} \rightarrow \mathbb{D}$ induces the quotient map $U \rightarrow G_p \backslash U$ for $G_p = \langle \gamma \rangle$.

b) all stabilizing subgroups are finite cyclic
c) fixed points of $G$ form a discrete subset
d) $G \backslash Y$ has a holomorphic structure as a Riemann surface s.t. the quotient map $Y \rightarrow G \backslash Y : z \mapsto Gz$ is holomorphic, ramified of multiplicity $n$ in fixed points of order $n$.

For now $Y = \mathbb{H}$ and $G$ discontinuous $\subset \text{Aut} \mathbb{H}$.

**Theorem 3.7** Aut $\mathbb{H} = \text{PSL}_2(\mathbb{R})$, the group of orientation preserving hyperbolic motions.

It is clear that Aut $\mathbb{H} \supseteq \text{PSL}_2 \mathbb{R}$, acting (simply) transitively on \{points\} and \{lines\}, suppose $\gamma \in \text{Aut} \mathbb{H}$ by combination with some $\mu \in \text{PSL}_2 \mathbb{R}$, suppose $\gamma(i) = i$, suppose $\gamma \in \text{Aut} \mathbb{D}$, and $\gamma(0) = 0$. Lemma (Schwarz): If holomorphic $\delta : \mathbb{D} \rightarrow \mathbb{D}$ has $\delta(0) = 0$, then

\[|\delta(z)| \leq |z| \quad \forall z \in \mathbb{D} \text{ with } "\leftarrow" \text{ iff}
\]

\[\delta(z) = \lambda z \text{ with } |\lambda| = 1.
\]

Hence, $\delta$ and its inverse mapping satisfy both $|\delta^{\pm 1}(z)| \leq |z|$, therefore $\delta(z) = \lambda z$ is a hyperbolic motion $\Rightarrow \Box$

**Theorem 3.8** $G \subset \text{PSL}_2(\mathbb{R})$ acts discontinuously on $\mathbb{H}$, iff $G$ is a discrete subgroup of $\text{PSL}_2(\mathbb{R})$.

### 3.2 Fuchsian Groups

There are two methods for the construction of Fuchsian groups:

- Arithmetic: Construct discrete groups of $\text{PSL}_2 \mathbb{R}$ by numbertheory, e.g. $\Gamma = \text{PSL}_2 \mathbb{Z}$ (modular group).
• Geometry (Poincaré): Start with a "suitable" hyperbolic polygon \( F \) (later serving as the fundamental domain for \( G \)), and generate \( G \) by side-pairing transformations.

Example: the "triangle groups" \( \langle l, m, n \rangle \) (drawing in \( \mathbb{D} \) instead of \( \mathbb{H} \)), \( l, m, n \in \mathbb{N} \setminus \{0\} \) or \( \infty \) with

\[
\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1.
\]

For example \( \langle 2, 3, \infty \rangle = \text{PSL}_2 \mathbb{Z} \) and

\[
\langle \infty, \infty, \infty \rangle = \Gamma(2) = \{ \gamma \in \Gamma = \text{PSL}_2 \mathbb{Z} | \gamma \equiv E \, \text{ mod } 2 \}
\]

Also \( \Gamma / \Gamma(2) = \text{PSL}_2(\mathbb{Z} / 2\mathbb{Z}) \cong S_3 \). Fact: there are 85 triangle groups which are "arithmetically defined" (Takeuchi ~ 1970).

\[
\gamma_0^l = 1 = \gamma_1^m = \gamma_\infty^n = \gamma_\infty \gamma_1 \gamma_0.
\]

**Theorem 3.9**  

a) These triangle groups \( \langle l, m, n \rangle = \langle \gamma_0, \gamma_1, \gamma_\infty \rangle \) are discontinuous on \( \mathbb{H} \) with \( F \) as "fundamental region" (i.e. \( F \) open and \( F \cap \gamma F = \emptyset \forall \gamma \in G - \{1\} \) and \( \bigcup_{\gamma \in G} \gamma F = \mathbb{H} \)) (hard work!)

\[ 
\vdots
\]

d) There is a meromorphic \( G \)-invariant function \( j : \mathbb{H} \to \hat{\mathbb{C}} \) \( (j(\gamma(z)) = j(z) \forall z \in \mathbb{H} \text{ and } \gamma \in G) \) mapping the two parts of \( F \) biholomorphically onto \( \mathbb{H} \) and \( -\mathbb{H} \), border edges onto \( \hat{\mathbb{R}} \) and the vertices onto \( 0, 1, \infty \), with multiplicities \( l, m, n \).
3.3 Problem

5 Let $G$ be a (possibly ramified) covering group of $X$ acting discontinuously on $Y$ with $X \cong G \setminus Y$ and let $N \triangleleft G$ normal subgroup, $X' := N \setminus Y$. Show that $G/N$ acts as group of automorphisms on $X'$ s.t.

$$(G/N) \setminus X' \cong X.$$ 

Give an example of a Riemann surface with an automorphism group $\text{PSL}_2(\mathbb{Z}/NZ)$, $N \in \mathbb{N}\setminus\{0\}$. 