Lecture 6

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2.3 More on Dessins

Isomorphism of dessins: If $\mathcal{B} = (G, x, y, E)$ and $\mathcal{B}' = (G', x', y', E')$ are bipartite maps (=dessins), then an isomorphism $i : \mathcal{B} \to \mathcal{B}'$ consists of a group-isomorphism $\theta : G \to G'$ sending x to x', y to y', and a bijection $\phi : E \to E'$ compatible with θ, i.e. $\phi(eg) = \phi(e)\theta(g)$ for all $e \in E$ and for all $g \in G$:

$$\begin{tikzcd} E \times G \arrow[r] \arrow[d, "\phi \,\times\, \theta"'] & E \arrow[d, "\phi"] \\ E' \times G' \arrow[r] & E' \end{tikzcd}$$

Theorem 2.2 *Every dessin $\mathcal{B}$ is isomorphic to $A \backslash \tilde{\mathcal{B}}$ for some regular dessin $\tilde{\mathcal{B}}$ and subgroup $A \subseteq \operatorname{Aut} \tilde{\mathcal{B}}$.*

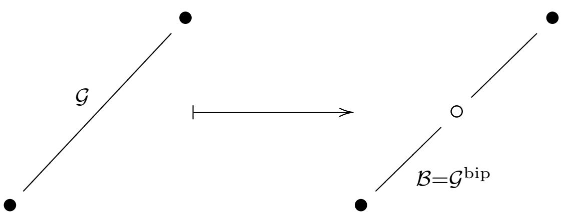
Proof. Take G to be the monodromy group of $\mathcal{B}$, and take $\tilde{\mathcal{B}}$ to be the dessin corresponding to the regular representation of G, so $\tilde{\mathcal{B}}$ is regular (Theorem 2.1). Take $A = \{ \lambda_g \mid g \in G_e \}$ for some $e \in E$; then orbits of A on E are just cosets $G_e g$ $(g \in G)$, so $A \backslash \tilde{\mathcal{B}} \cong \mathcal{B}$. $\square$

Call $\tilde{\mathcal{B}}$ the canonical regular cover of $\mathcal{B}$.

Exercise 2.4 *Let $\mathcal{B}$ consist of a path of N edges, alternately white, black, white, etc.* $\bullet$ —— $\circ$ —— $\bullet$ —— $\circ$ —— $\cdots$ *Find G, C, $\tilde{\mathcal{B}}$ and A for this dessin.*

What about embeddings of graphs $\mathcal{G}$ which are not necessarily bipartite, e.g. the tetrahedron or octahedron?

Convert $\mathcal{G}$ into a bipartite graph by regarding the vertices of $\mathcal{G}$ as black vertices, and placing a white vertex in each edge of $\mathcal{G}$.



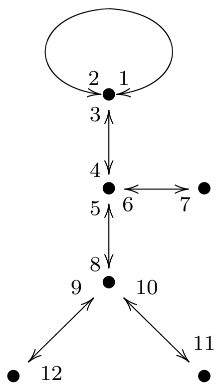
This gives a bipartite graph $\mathcal{G}^{\mathrm{bip}}$. Any embedding of $\mathcal{G}$ in a surface gives a bipartite map $\mathcal{B}$. The edges of $\mathcal{B}$ correspond to the directed edges (=darts) of $\mathcal{G}$. The rotations x and y of the set E of edges of $\mathcal{B}$ correspond to rotations x and y of the set Ω of darts of $\mathcal{G}$. So x rotates darts α around their incident vertices following the orientation of the surface amd y reverses the direction of each dart, so $y^2 = 1$.

We can define an algebraic map (not necessarily bipartite) to be a 4-tuple (G, x, y, Ω) where $G = \langle x, y \rangle$ is a transitive permutation group acting on Ω, with $y^2 = 1$. As before, we can identify the vertices, edges and faces with cycles of x, y and xy on Ω, incidence given by non-empty intersection.

The algebraic theory is similar to that for bipartite maps.

2.4 Example

$\mathcal{M}$, Monsieur Mathieu:



Here $|\Omega| = 12$. So

$$x = (1\,2\,3)(4\,5\,6)(7)(8\,9\,10)(11)(12)$$

and

$$y = (1\,2)(3\,4)(5\,8)(6\,7)(9\,12)(10\,11).$$

Now $G = \langle x, y \rangle$. GAP $\Rightarrow |G| = 95040, G \cong M_{12}$.

Finite simple groups (classified $\sim$ 1980): C_p, A_n, where $(n \geq 5)$, groups of Lie type, e.g. $\mathrm{PSL}_2(\mathbb{F}_q)$, 26 sporadic groups, e.g. Mathieu group M_n where $n = 11, 12, 22, 23, 24$. In this example, $G_\alpha \cong M_{11}$ for $\alpha \in \Omega$. $\mathcal{M}$ has genus 0, and type $(3, 2, 11)$. The corresponding bipartite map $\mathcal{B}$ has canonical regular cover $\tilde{\mathcal{B}}$ of type $(3, 2, 11)$ and genus $g = 3601$ (see Exercise 2.3), $\mathrm{Aut}\,\tilde{\mathcal{B}} \cong M_{12}$.

By Belyi's theorem $\tilde{\mathcal{B}}$ corresponds to an algebraic curve defined over an algebraic number field. The field of definition is $\mathbb{Q}(\sqrt{-11})$. This has Galois group isomorphic to C_2, generated by complex conjugation. Applying this to the coefficients of the algebraic curve and the Belyi function, we get the mirror image of $\mathcal{M}$, $\bar{\mathcal{M}}$. Later we will see more interesting and less obvious actions of Galois groups of maps.

3 Galois Theory

3.1 Basic Galois Theory

Every field F has an algebraic closure $\bar{F}$, a minimal extention field of F over which every $f \in F[x]$ splits into linear factors. This field $\bar{F}$ is:

- unique up to isomorphisms fixing F,
- an algebraic extension of F, i.e. every $\alpha \in \bar{F}$ is a root of some non-zero $f \in F[x]$, or equivalently $|F(\alpha) : F| < \infty$.

Important case:

$$\bar{\mathbb{Q}} := \{\, \alpha \in \mathbb{C} \,|\, f(\alpha) = 0 \text{ for some non-zero } f \in \mathbb{Q}[x] \,\}$$

the field of algebraic numbers. Motivation: Belyi's Theorem.

A field extension $K \supseteq F$ is *normal* (or *Galois*) if every embedding $e : K \hookrightarrow \bar{F}$ (fixing F) satisfies $e(K) = K$.

(Strictly speaking, "Galois = normal and separable", where "separable" means that irreducible polynomials don't have repeated roots; all fields of characteristic 0 are separable, so we'll ignore this point by assuming that $\operatorname{char} F = 0$ for all fields F mentioned.)

Example 3.1 $F = \mathbb{Q}$, $K = \mathbb{Q}(\zeta_n)$ *the* n^{th} *cyclotomic field,* $\zeta_n = \exp(\frac{2\pi i}{n})$. *Any embedding* $e : \mathcal{K} \hookrightarrow \bar{\mathbb{Q}}$ *sends* ζ_n *to some* $\zeta_n^j \in K$, *so* $e(K) = K$. *This is Galois extension.*

Example 3.2 $F = \mathbb{Q}$, $K = \mathbb{Q}(\alpha)$, $\alpha = 2^{1/3} \in \mathbb{R}$. *There is an embedding* $e : K \hookrightarrow \bar{\mathbb{Q}}$ *sending* α *to* $\alpha\zeta_3 \notin K$. *This extension is not Galois.*

Theorem 3.1 $K \supseteq F$ *is a finite Galois extension if and only if* K *is the splitting field of some* $f \in F[x]$.

The Galois group $\operatorname{Gal} K$ of a field K is the group of all field automorphisms of K. If $H \leq \operatorname{Gal} K$, then $\operatorname{fix} H$ is the subfield fixed pointwise by H. If $F \subseteq K$ then $\operatorname{Gal} K/F$ is the subgroup of $\operatorname{Gal} K$ fixing F pointwise.

In theorem 3.1 $G = \operatorname{Gal} K/F$ permutes the roots of f faithfully so we can embed G in S_n, $n = \deg(f) =$"no. of roots of f", and $|G| = |K : F|$.

Example 3.3 *$K = \mathbb{Q}(\alpha, \zeta_3)$, $\alpha = 2^{1/3} \in \mathbb{R}$ as before, $F = \mathbb{Q}$. K is the splitting field of $f(x) = x^3 - 2$. Degree is $|K : F| = 6$, basis $1, \alpha, \alpha^2, \zeta_3, \alpha\zeta_3, \alpha^2\zeta_3$. f has three roots $\alpha_j = \alpha\zeta_3^j$ ($j = 0, 1, 2$) permuted faithfully by $G = \operatorname{Gal} K/F$, so $G \hookrightarrow S_3$. Since $|G| = |K : F| = 6$ and $|S_3| = 6$, $G \cong S_3$.*

Theorem 3.2 (Fundamental Theorem of Galois Theory) *Let $K \supseteq F$ be a finite Galois extension, $G = \operatorname{Gal} K/F$. There is an order-reversing bijection $L \mapsto H = \operatorname{Gal} K/L$ between fields L such that $K \supseteq L \supseteq F$, and subgroups $H \leq G$. The inverse sends each H to $L = \operatorname{fix} H$. We have $|K : L| = |H|$ and $|L : F| = |G : H|$. $L \supseteq F$ is Galois iff $H \trianglelefteq G$, in which case $\operatorname{Gal} L/F \cong G/H$.*

$$\begin{tikzcd}
K \arrow[d, dash] \arrow[ddr, dashed, leftrightarrow] & G \arrow[d, dash] \arrow[ddl, dashed, leftrightarrow] \\
L \arrow[r, leftrightarrow] \arrow[d, dash] & H \arrow[d, dash] \\
F & 1
\end{tikzcd}$$

In example 1,

$$\operatorname{Gal} \mathbb{Q}(\zeta_n)/\mathbb{Q} = \{\, \theta_j : \zeta_n \mapsto \zeta_n^j \mid (j, n) = 1 \,\} \cong U_n = \mathbb{Z}_n^*,$$

the group of units $\mod n$. This is abelian, so all subfields of $\mathbb{Q}(\zeta_n)$ are Galois over $\mathbb{Q}$.

In example 3, $S_3 \rhd A_3 \cong C_3$, and the field L corresponding to $H = A_3$ is the Galois extension $\mathbb{Q}(\zeta_3)$ of $\mathbb{Q}$. The subfield $L = \mathbb{Q}(\alpha)$ corresponds to a non-normal subgroup of G

Exercise 3.1 *Find the splitting field K of $x^n - 2$, describe the Galois group of K, and find the subgroups fixing $2^{1/n} \in \mathbb{R}$ and ζ_n.*

3.2 The Absolute Galois Group

The *absolute Galois group* of a field F is $\operatorname{Gal} \bar{F}/F$. The *absolute Galois group* is $\operatorname{Gal} \bar{\mathbb{Q}}/\mathbb{Q}$, denoted by $\underline{\underline{G}}$. Let $\mathcal{K}$ denote the set of all finite Galois extensions K of $\mathbb{Q}$, and let $G_K = \operatorname{Gal} K/\mathbb{Q}$, a finite group of order $|K : \mathbb{Q}|$.

Theorem 3.3 *(i) $\bar{\mathbb{Q}}$ is the union of all the fields $K \in \mathcal{K}$*

(ii) Each $K \in \mathcal{K}$ is invariant under $\underline{\underline{G}}$.

Proof.

(i) Each $K \in \mathcal{K}$ is a finite extension of $\mathbb{Q}$, so if $\alpha \in K$ then $|\mathbb{Q}(\alpha) : \mathbb{Q}| \leq |K : Q| < \infty$, so $\alpha \in \bar{\mathbb{Q}}$. Conversely, if $\alpha \in \bar{\mathbb{Q}}$ then $f(\alpha) = 0$ for some non-zero $f \in \mathbb{Q}[x]$, and $\alpha \in K =$"splitting field of f".

(ii) Follows by definition of "Galois".

□

Thus each $g \in \underline{\underline{G}}$ is uniquely determined by its restrictions $g_K \in G_K$ to the fields $K \in \mathcal{K}$. If $K \supseteq L$ where $K, L \in \mathcal{K}$ then L is invariant under G_K so there is a restriction homomorphism $\rho_{K,L} : G_K \to G_L$ sending g_K to g_L, i.e.

$$\rho_{K,L}(g_K) = g_L$$

whenever $K \supseteq L$. Conversely if we have elements $g_K \in G_K$ for each $K \in \mathcal{K}$, with $\rho_{K,L}(g_K) = g_L$ whenever $K \supseteq L$, we can define $g \in \underline{\underline{G}}$ by $g(\alpha) = g_K(\alpha)$ where $\alpha \in K \in \mathcal{K}$. (Check independence of K.) We can therefore identify $\underline{\underline{G}} = \mathrm{Gal}\,\bar{\mathbb{Q}}$ with the group

$$\{\, (g_K) \in \Pi := \Pi_{K \in \mathcal{K}} G_K \mid \rho_{K,L}(g_K) = g_L \text{ whenever } K \supseteq L \text{ in } \mathcal{K} \,\},$$

the subgroup of the cartesian product Π consisting of elements whose coordinates are compatible with the $\rho_{K,L}$'s.

This is the projective limit $\varprojlim G_K$ of the finite groups G_K and homomorphisms $\rho_{K,L}$, a profinite group.

Exercise 3.2 *Show that* $\bigcup_{n \geq 1} \mathbb{Q}(\zeta_n)$ *is a subfield of* $\bar{\mathbb{Q}}$*, and describe its Galois group.*

Exercise 3.3 *What are the cardinalities of* $\bar{\mathbb{Q}}$ *and* $\underline{\underline{G}}$*?*

To get a bijection between fields and groups, we need some topology:

Put the discrete topology on each G_K ($K \in \mathcal{K}$), so all subsets are open and closed. This induces a product topology on Π, the weakest such that the projections $\Pi \to G_K$ are continuous. $\underline{\underline{G}} \hookrightarrow \Pi$, so $\underline{\underline{G}}$ inherits a topology from Π, the *Krull topology.* (Intuitively, elements of $\underline{\underline{G}}$ are "close together" if they agree on a large subfield of $\bar{\mathbb{Q}}$.) Multiplication and inversion are continuous in each G_K, and hence also in Π and $\underline{\underline{G}}$, so these are topological groups.

Exercise 3.4 *Show that* $\underline{\underline{G}}$ *is a closed subgroup of* Π*, and both* Π *and* $\underline{\underline{G}}$ *are compact Hausdorff spaces.*

Warning: $\underline{\underline{G}}$ is topologially unpleasant: homeomorphic to a Cantor set.

The Fundamental Theorem (3.1) extends to the extension $\bar{\mathbb{Q}} \supseteq \mathbb{Q}$ provided we rectrict the bijection to the closed subgroups of $\underline{\underline{G}}$, *not* all subgroups.

Exercise 3.5 *In any topological group, every open subgroup is closed, and every closed subgroup of finite index is open.*