Lecture 7

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Theorem 3.9 *a) Fuchsian triangle groups $\mathcal{G}$ are discontinuous on $\mathbb{H}$ with F as "fundamental domain".*

b) $\mathcal{G}$ is generated by γ_0, γ_1, γ_∞ (by any two of them)

c) $\mathcal{G}$ is presented by $\langle \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0^l = \gamma_1^m = \gamma_\infty^n = 1 = \gamma_\infty\gamma_1\gamma_0 \rangle$

d) There is a meromorphic $\mathcal{G}$-invariant ($\mathcal{G}$-"automorphic") function $j : \mathbb{H} \to \hat{\mathbb{C}}$ mapping the two (open) parts of F onto $\mathbb{H}$ and $-\mathbb{H}$, border edges on $\hat{\mathbb{R}}$, vertices to $0, 1, \infty$ with ramification multiplicities l, m, n.

e) j provides an identification of $\mathcal{G}\backslash\mathbb{H}$ with $\hat{\mathbb{C}}$ (in case that l, m, n finite)

Remark: In case of cusps, omit these points! I.e. for $\mathcal{G} = \langle \infty, \infty, \infty \rangle \cong \Gamma(2)$, $j : \mathbb{H} \to \hat{\mathbb{C}} - \{ 0, 1, \infty \}$ universal covering map!

3.4 Problems

6 Show that $\langle 2, 2, n \rangle = \mathcal{G}$ is a "spherical" trianglegroup and

$$j(z) = z^n + \frac{1}{z^n}.$$

3.5 Remarks

Triangle groups are (the only) "rigid" Fuchsian groups, i.e. uniquely determined by their presentation upto conjugation in $\mathrm{PSL}_2(\mathbb{R})$.

There is a bijection between $\{ \mathcal{G} -$ automorphic functions on $\mathbb{H} \}$ and $\{$ meromorphic functions on $\mathcal{G}\backslash\mathbb{H} \}$.

3.6 More General Facts about Fuchsian Groups

Theorem 3.10 *a) Let $p \in \mathbb{H}$ be a non-fixed point for $\mathcal{G}$ and let d be the hyperbolic distance on $\mathbb{H}$. Then (Dirichlet)*

$$F := \{ z \in \mathbb{H} \mid d(p, z) < d(\gamma(p), z) \ \forall \gamma \in \mathcal{G} - \{\mathrm{id}\} \} \neq \emptyset$$

is a fundamental domain for $\mathcal{G}$ bounded by side-edges

$$l_\sigma := \{ z \in \mathbb{H} \mid d(p, z) = d(\sigma(p), z), \ \sigma \in \mathcal{G} - \{\mathrm{id}\} \\ d(p, z) \le d(z, \gamma(z)) \ \forall \gamma \in \mathcal{G} - \{\mathrm{id}, \sigma\} \}.$$

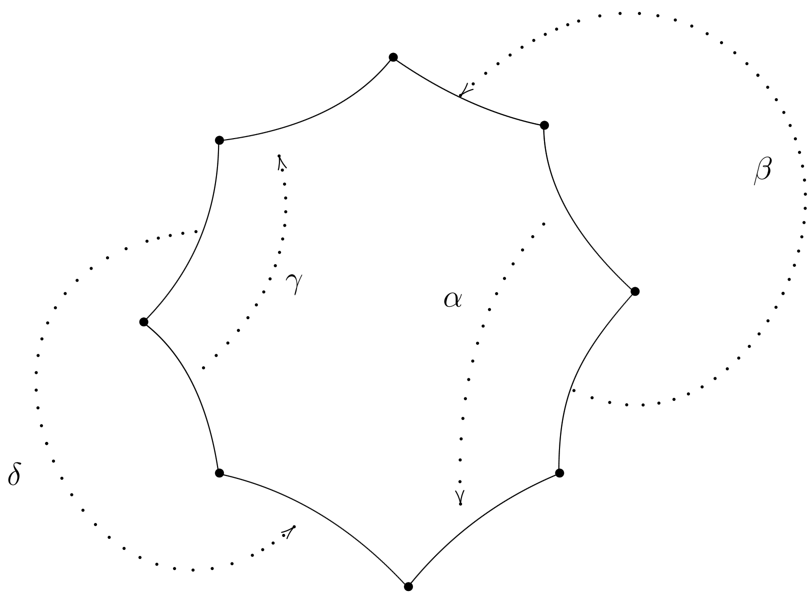
b) For all compact $C \subset \mathbb{H}$, $l_\sigma \cap C \neq \emptyset$ for finitely many $\sigma \in \mathcal{G}$ only. $\bar{F}$ compact $\Rightarrow$ F is a finite convex polygon bounded by finitely many side edges, $\bar{F}$ compact $\subset \mathbb{H} \Leftrightarrow X = \mathcal{G}\backslash\mathbb{H}$ is a compact Riemann surface.

c) $\mathcal{G}$ is generated (finitely in the "cocompact" case) by "side-pairing" transformations $\sigma \in \mathcal{G}$ sending $l_{\sigma^{-1}}$ to l_σ, sending F to a neighbour σF with common side l_σ.

d) Loops around vertices of F $\rightsquigarrow$ relations between these generators $\rightsquigarrow$ presentation of $\mathcal{G}$.

e) (Poincaré) If F is an $\mathbb{H}$-polygon with side-pairings and some condition on the angles guaranteeing that locally around F, the images γF have no overlapping $\Rightarrow$ the plane is covered by $\mathcal{G}F$ without overlappings and with $\mathbb{H} = \mathcal{G}\bar{F}$, $\mathcal{G} = \langle$ side-pairings$\rangle$.

Example (in $\mathbb{D}$): Let F be an 8-sided polygon with side-pairings as indicated below,



$\sum$ all angles $= 2\pi \Rightarrow$

$$\mathcal{G} = \langle\, \alpha, \beta, \gamma, \delta \,|\, \alpha\beta\alpha^{-1}\beta^{-1}\gamma\delta\gamma^{-1}\delta^{-1} = 1 \,\rangle$$

is a Fuchsian group with $\mathcal{G}\backslash\mathbb{H} = X$ compact of $g = 2$. This $\mathcal{G}$ is not rigid! $\mathcal{G}\backslash\mathbb{H}$ has six real free parametres $\Rightarrow$ "Teichmüller space".

Theorem 3.11 *a) Suppose $\mathcal{G} \subset \Delta$ are Fuchsian, $(\Delta : \mathcal{G}) < \infty$ with $\Delta = \dot{\bigcup}_k \mathcal{G}\gamma_k$, Δ has F_Δ as a fundamental domain. Then $F_\mathcal{G} := \bigcup_k \gamma_k F_\Delta$ is a fundamental domain for $\mathcal{G}$ ($\Rightarrow$ inducing a triangulation of $X = \mathcal{G}\backslash\mathbb{H}$ if Δ is a triangle group).*

b) Let X, X' be Riemann surfaces with surface (universal covering) groups $\mathcal{G}, \mathcal{G}' \subset \operatorname{Aut}\mathbb{H} = \mathrm{PSL}_2\,\mathbb{R}$. Then $X \cong X'$ iff $\mathcal{G}$ and $\mathcal{G}'$ are conjugate in $\mathrm{PSL}_2\,\mathbb{R}$.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\gamma\in\operatorname{Aut}\mathbb{H}} & \mathbb{H} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\cong} & X' \end{array}$$

(γ well defined $\Leftrightarrow$ induces a conjugation $\mathcal{G} \to \mathcal{G}'$)

Recall that X compact Riemann surface is a smooth projective algebraic curve given by some equations. In case $g > 1$, $X = \mathcal{G}\backslash\mathbb{H}$ with some Fuchsian group $\mathcal{G}$. How are the equations determined by $\mathcal{G}$ and conversely?

Theorem 3.12 *Suppose X is a (compact) projective smooth algebraic curve. It has a Belyi function $\beta : X \to \hat{\mathbb{C}}$ (i.e. can be defined over $\bar{\mathbb{Q}}$) $\Leftrightarrow$ there is a triangle group $\Delta = \langle\, l, m, n \,\rangle$ (cocompact) and a finite index subgroup $\mathcal{G} \subseteq \Delta$ s.t. $X \cong \mathcal{G}\backslash\mathbb{H}$*

Proof.

"$\Leftarrow$": If $X \cong \mathcal{G}\backslash\mathbb{H}$, $\mathcal{G} \subseteq \Delta$, then $j : \mathbb{H} \to \hat{\mathbb{C}}$ with the j-function for $\Delta = \langle\, l, m, n \,\rangle$ induces a well-defined meromorphic mapping $\beta : \mathcal{G}z \mapsto j(z)$ ramified only in points $\mathcal{G}\Delta p_0, \mathcal{G}\Delta p_1, \mathcal{G}\Delta p_\infty \in X = \mathcal{G}\backslash\mathbb{H}$ (p_i fixed under γ_i), therefore a Belyi function; the dessin given by the Δ-tesselation on the upper half-plane $\mathbb{H}$, take the quotient by $\mathcal{G}$.

"$\Rightarrow$": Start with a Belyi function $\beta : X \to \hat{\mathbb{C}}$ s.t. least common multiple (lcm) of all multiplicities above 0, (1,∞) is l, (m,n). (Any common multiple does as well!) $\Delta = \langle\, l, m, n \,\rangle \subset \mathrm{PSL}_2\,\mathbb{R}$ and its j-function $\Rightarrow \beta^{-1}$ is only locally biholomorphic outside $0, 1, \infty$, but $\beta^{-1} \circ j$ is everywhere locally well-defined, holomorphically, so

$$\begin{array}{ccccccc} w & & \mathbb{H} & & & w & \\ \downarrow & & {\scriptstyle h}\,\vdots & \searrow^{j} & & & \searrow \\ w^{l/l'} & & X & \xrightarrow{\beta} & \hat{\mathbb{C}} & & w^l \\ \| & & & & & & \\ z & \longmapsto & z^{l'} & & & & \end{array}$$

commutes. $\mathbb{H}$ simply connected, so (by the monodromy theorem) $\beta^{-1} \circ j$ can be defined globally as holomorphic map h

$$h(z) = h(z') \Rightarrow z \in \Delta z'$$

So $X \cong \mathcal{G} \backslash \mathbb{H}$, where $\mathcal{G}$ is defined as $\{ \gamma \in \Delta \,|\, h(z) = h(\gamma z) \text{ for all } z \in \mathbb{H} \}$.

□

3.7 Remarks

- In general, $\mathcal{G}$ is not the (unique) surface group for X, because it can have torsion. But if $l' = l$ etc., i.e. if β has the same multiplicity l (m, n) in all zeros (1-points, poles), then h is the universal covering map, $\mathcal{G}$ is the surface group of X. This occurs precisely, if the dessin for β is "uniform" ($\Leftarrow$ regular dessins).
- $\deg \beta = (\Delta : \mathcal{G})$.