Lecture 8

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4 From Dessins to Holomorphic Structures

4.1 Coverings

Let $\mathcal{B} = (G, x, y, E)$ and $\mathcal{B}' = (G', x', y', E')$ be algebraic bipartite maps. A *morphism* $\gamma : \mathcal{B} \to \mathcal{B}'$ or *covering* consists of a group-homomorphism $\theta : G \to G'$ and a function $\phi : E \to E'$ such that $x \mapsto x'$ and $y \mapsto y'$ under θ, and $\phi(eg) = \phi(e)\theta(g)$ for all $e \in E$, and for $g = x, y$ (equivalently for all $g \in G$).

Example: $\mathcal{B}_1 \to \mathcal{B}_2 = C_2 \backslash \mathcal{B}_1$ in lecture 2.

More generally, $\mathcal{B} \to A\backslash\mathcal{B} = \mathcal{B}'$, where $A \leq \operatorname{Aut}\mathcal{B}$ with G', x', y' the actions of G, x and y on the orbits of A. Coverings induced by automorphisms in this way are *regular*, or *normal*.

Exercise 4.1. *Show that θ and ϕ must be epimorphisms.*

γ is an isomorphism, iff θ and ϕ are bijections, and then an automorphism if $\mathcal{B} = \mathcal{B}'$.

Exercise 2.2: $\operatorname{Aut}\mathcal{B} = C(G) \cong N_G(G_e)/G_e$. Algebraic bipartite maps form a category.

The topological analogue of a morphism γ is a branched covering $X \to X'$ of surfaces, preserving orientation, with black vertices, white vertices, edges and faces on X' lifting to the same on X, and branching only at vertices or face-centers. We have a category of topological bipartite maps, and lecture 2 described a functor from these to algebraic bipartite maps. We can easily reverse this process, but with more work we can obtain holomorphic, rather than topological structures from algebraic bipartite maps.

4.2 Triangle Groups and Bipartite Maps

Consider algebraic bipartite maps of a given type (l, m, n), so in G we have $x^l = y^m = z^n = xyz = 1$. Consider the (abstract) group

$$\Delta = \Delta(l, m, n) = \langle\, X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \,\rangle$$

G is a quotient of Δ by $X \mapsto x$, etc. Then $\Delta \to G \to \mathrm{Sym}(E)$ gives a transitive action of Δ on the edge set E of $\mathcal{B}$. Bipartite maps of type $(l, m, n) \leftrightarrow$ "transitive actions of Δ". (Warning: actions of Δ can give maps of type (l', m', n') where $l'|l$, etc.) These actions correspond to conjugacy classes of subgroups $\Delta_e \leq \Delta$ $(e \in E)$. $\mathcal{B}$ is finite (compact) $\Leftrightarrow |\Delta : \Delta_e| < \infty$. Coverings $\mathcal{B} \to \mathcal{B}'$ correspond to inclusions $\Delta_e \leq \Delta_{e'}$ (easy exercise). Regular coverings correspond to normal inclusions. $\mathrm{Aut}\,\mathcal{B} \cong N_\Delta(\Delta_e)/\Delta_e$ (exercise 2.2) Theorem 2.1 and exercise 2.2 give

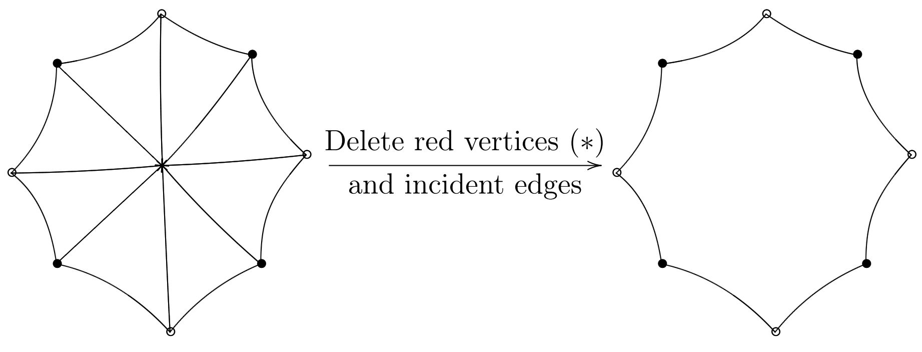
Theorem 4.1. *$\mathcal{B}$ is regular if and only if $\Delta_e \trianglelefteq \Delta$, in which case*

$$G \cong \mathrm{Aut}\,\mathcal{B} \cong \Delta/\Delta_e.$$

Example 4.1. Let $\mathcal{B}$ correspond to the regular representation of $G = C_n \times C_n = \langle\, x, y \,|\, x^n = y^n = 1, xy = yx \,\rangle$. Then xy has order n, so the type is (n, n, n). Take $\Delta = \Delta(n, n, n)$, $\Delta_e = \mathrm{Ker}(\Delta \to G) \trianglelefteq \Delta$. G is abelian, so $\Delta_e \geq \Delta' =$ "commutator subgroup of Δ". Both have index n^2 in Δ, so $\Delta_e = \Delta'$. Here $G \cong \mathrm{Aut}\,\mathcal{B} \cong \Delta/\Delta_e = \Delta^{\mathrm{ab}}$.

The triangle group of type (l, m, n) has the same presentation as Δ (generators $\gamma_0, \gamma_1, \gamma_\infty$ in Jürgen's lectures), so identify Δ with this group, $X, Y, Z =$ "rotations through $\frac{2\pi}{l}, \frac{2\pi}{m}, \frac{2\pi}{n}$ about the vertices of a triangle T with internal angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$". Assume that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ (typical case); if not, replace $\mathbb{H}$ with $\mathbb{C}$ or $\hat{\mathbb{C}}$. $\mathbb{H}$ is tesselated by the images of T under the extended triangle group $\Delta[l, m, n]$ generated by reflections in the sides of T, and $\Delta = \Delta(l, m, n)$ is the even subgroup of index 2, preserving orientation.

We can colour the vertices black, white or red as they are images of the vertices of T fixed by X, Y or Z. Every triangle has one vertex of each colour. Their valencies are $2l, 2m, 2n$ respectively.



This gives a bipartite map of type (l, m, n) on $\mathbb{H}$. This is the universal bipartite map $\mathcal{B}_\infty(l, m, n)$ of type (l, m, n). It is a regular map, with $\mathrm{Aut}\,\mathcal{B}_\infty(l, m, n) = \Delta(l, m, n)$, edge-stabiliser $\Delta_e = 1$.

Theorem 4.2. *Every bipartite map $\mathcal{B}$ of type (l,m,n) is isomorphic to a quotient $A\backslash\mathcal{B}_\infty(l,m,n)$ of $\mathcal{B}_\infty(l,m,n)$ by a subgroup $A \le \operatorname{Aut}\mathcal{B}_\infty(l,m,n)$.*

Proof. Take A to consist of the automorphisms of $\mathcal{B}_\infty(l,m,n)$ induced by the subgroup Δ_e of Δ, and check that $\mathcal{B} \cong A\backslash\mathcal{B}_\infty(l,m,n)$. □

4.3 Holomorphic Structures

$A\backslash\mathcal{B}_\infty(l,m,n)$ has extra holomorphic structure, so denote it by $\mathcal{B}^{\text{hol}}$. $\mathbb{H}$ is a Riemann surface, and Δ_e acts as a discontinuous group of automorphisms of $\mathbb{H}$ (since Δ does), so $\mathcal{B}^{\text{hol}}$ is on a Riemann surface $X = A\backslash\mathbb{H}$. Coverings $\mathcal{B} \to \mathcal{B}'$ of bipartite maps correspond to inclusions $\Delta_e \le \Delta_{e'}$ in Δ, so these induce branched coverings $X \to X'$ of Riemann surfaces. In particular, if we take $\Delta_{e'} = \Delta$, so $|E'| = 1$ corresponding to the trivial bipartite map with one edge, we get a covering $X \to X' = \hat{\mathbb{C}}$ branched only over the vertices 0 and 1, and the face-centre at ∞. This is a Belyi function (provided X is compact, i.e. $\mathcal{B}$ is finite). Then Belyi's Theorem gives:

Theorem 4.3. *If $\mathcal{B}$ is a finite algebraic map, then the Riemann surface X underlying $\mathcal{B}^{hol}$ is defined, as a smooth projective algebraic curve, over the field $\bar{\mathbb{Q}}$ of algebraic numbers.*

Example 4.2 (Example 4.1 revisited). If $\mathcal{B}$ is as in example 4.1, the Riemann surface X uniformised by Δ' (="commutator subgroup of $\Delta = \Delta(n,n,n)$") is the n^{th} degree Fermat curve $F = F_n$ with affine equation $x^n + y^n = 1$, with Belyi function $\beta : (x,y) \mapsto x^n$. The black vertices are at $(0,\zeta_n^j)$ $j = 0,1,\ldots,n-1$, and the white vertices are at $(\zeta_n^k,0)$ $k = 0,1,\ldots,n-1$. The edges (given by $\beta^{-1}([0,1])$) between $v_j = (0,\zeta_n^j)$ and $w_k = (\zeta_n^k,0)$ are given by $(r\zeta_n^k, s\zeta_n^j)$ where $r,s \in [0,1]$ and $r^n + s^n = 1$.

In general,

$$\begin{aligned}\operatorname{Aut}\mathcal{B} \cong \operatorname{Aut}\mathcal{B}^{\text{hol}} &\cong N_\Delta(\Delta_e)/\Delta_e \\ &\le N_{\mathrm{PSL}_2\mathbb{R}}(\Delta_e)/\Delta_e \qquad (\text{since } \Delta \le \mathrm{PSL}_2\mathbb{R} \\ &\cong \operatorname{Aut} X.\end{aligned}$$

Thus automorphisms of $\mathcal{B}$ act as automorphisms of the Riemann surface X (equivalently, of the algebraic curve).

Example 4.3 (=Examples 1 and 2 revisited). If $\mathcal{B}$ is as in example 4.1 and 4.2, then $\operatorname{Aut}\mathcal{B} \cong C_n \times C_n$, and this acts on X by multiplying x and y

independently by n^{th} roots of 1. In this case, $\operatorname{Aut}\mathcal{B} \neq \operatorname{Aut} X$, since $\operatorname{Aut} X$ is a semidirect product $(C_n \times C_n) \rtimes S_3$ of $\operatorname{Aut}\mathcal{B}$ by a complement S_3. The extra S_3 comes from permuting the 3 vertex-colours, or alternatively write X in projective form as $x^n + y^n + z^n = 0$, and let S_3 permute the coordinates.

Exercise 4.2. *Explain example 4.3 by describing* $N_{\mathrm{PSL}_2\mathbb{R}}(\Delta_e)$.

4.4 Non-cocompact Trianle Groups

Suppose we want to consider all bipartite maps $\mathcal{B}$ of type $(3, 2, n)$ without restricting n. We take

$$\begin{aligned}\Delta = \Delta(3, 2, \infty) &= \langle\, X, Y, Z \mid X^3 = Y^2 = Z^\infty = XYZ = 1 \,\rangle \\ &= \langle\, X, Y \mid X^3 = Y^2 = 1 \,\rangle \qquad \text{eliminating } Z = (XY)^{-1} \\ &\cong C_3 * C_2.\end{aligned}$$

The algebraic theory works as before. Geometrically, we take T to have a black vertex at i (angle $\frac{\pi}{2}$) and white vertex at ζ_3 (angle $\frac{\pi}{3}$), and a red vertex at ∞ on $\partial\mathbb{H}$ (angle $\frac{\pi}{\infty} = 0$). Reflections in the sides of T generate $\Delta[3, 2, \infty]$, the images of T tesselate $\mathbb{H}$, with vertices at the images of ∞.

Exercise 4.3. *Show that* $\Delta[3, 2, \infty] = \mathrm{PGL}_2(\mathbb{Z})$*, consisting of the transformations*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \qquad a, \ldots, d \in \mathbb{Z},\; ad - bc = 1$$

or

$$\tau \mapsto \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \qquad a, \ldots, d \in \mathbb{Z},\; ad - bc = -1.$$

The first type form the even subgroup $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

The orbit of ∞ under Γ is $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, so this is the set of red vertices. Deleting the red vertices and their incident edges, we get a bipartite map $\mathcal{B}_\infty(3, 2, \infty)$ of type $(3, 2, \infty)$. If Δ_e is a subgroup of finite index in $\Delta = \Gamma$, then $\Delta_e \backslash \mathbb{H}$ is a compact Riemann surface minus finitely many points, one for each orbit of Δ_e on $\mathbb{P}^1(\mathbb{Q})$

To deal with bipartite maps $\mathcal{B}$ of all possible types, use $\Delta(\infty, \infty, \infty) = \Gamma(2)$, congruence subgroup of level 2 in Γ. Here T has 3 vertices on $\partial\mathbb{H}$, at $0, 1$ and ∞. $\Gamma(2)$ is the even subgroup of $\Delta[\infty, \infty, \infty] =$ "group generated by reflections in the sides of T". Images of T tesselate $\mathbb{H}$, vertices are elements $\frac{p}{q} \in \mathbb{P}^1(\mathbb{Q})$, coloured black, white, red, as p is even and q is odd, or p and q are both odd, or p is odd and q is even (orbits of $\Gamma(2)$, see exercise 1.3). Deleting red vertices and incident edges gives $\mathcal{B}_\infty(\infty, \infty, \infty) = \mathcal{B}_\infty$, the universal bipartite map. Every $\mathcal{B}$ is a quotient of $\mathcal{B}_\infty$.

Exercise 4.4. *Draw* $\mathcal{B}_\infty$*!*