Lecture 9

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5 Quasiplatonic Surfaces, and Automorphisms

5.1 Definitions and Properties

Any compact Riemann surface X of genus $g > 1$ can be uniformised by an essentially unique (upto conjugation by isometries) torsion-free Fuchsian group K ($\cong \pi_1 X$). Isomorphisms $X \to X'$ are induced by conjugating isometries of $\mathbb{H}$ taking K to K'. Taking $X = X'$ we see that automorphisms of X are induced by isometries normalising K. Since K acts trivially on $K\backslash\mathbb{H}$, we get

$$\mathrm{Aut}\, X \cong N(K)/K$$

where N denotes the normalizer in $\mathrm{PSL}_2\,\mathbb{R}$. (If $g = 1$, replace $\mathbb{H}$ with $\mathbb{C}$, replace K with a lattice Λ unique up to similarity — see the Elliptic Curves lecture.) We say that X (compact, of genus $g > 1$), is *quasiplatonic* if X is uniformised by a subgroup K as above, with K normally contained in a triangle group.

Theorem 5.1. *If X is a compact Riemann surface of genus $g > 1$, the following are equivalent:*

a) X is quasiplatonic,

b) $N(K)$ is a triangle group (K as above)

c) X has a Belyi function $\beta : X \to \hat{\mathbb{C}}$ which is a regular covering,

d) X corresponds to a regular dessin.

Proof.

a) $\Rightarrow$ b): $N(K)$ is a Fuchsian group (since K is) and it contains a triangle group. Any Fuchsian group containing a triangle group must be a triangle group (by Teichmüller theory — triangle groups are the only rigid Fuchsian groups).

b) $\Rightarrow$ c): Inclusion $K \leq N(K)$ induces a Belyi function

$$X \cong K \backslash \mathbb{H} \to N(K) \backslash \mathbb{H} \cong \hat{\mathbb{C}}.$$

Since $K \trianglelefteq N(K)$, this is a regular covering.

c) $\Rightarrow$ d): Use β to lift the trivial dessin ($\underset{0}{\circ}$ —— $\underset{1}{\bullet}$) on $\hat{\mathbb{C}}$ to X, and since β is regular we get a regular dessin on X.

d) $\Rightarrow$ a): If X corresponds to a regular dessin $\mathcal{B}$, then K is normal in the corresponding triangle group.

□

Example 5.1. The n^{th} degree Fermat curve ($n > 3$) corresponds to a regular dessin, and is uniformised by the commutator subgroup of $\Delta(n, n, n)$ which is normal.

Exercise 5.1. *For genus $g = 1$, what are the analogues of the quasiplatonic surfaces?*

One can characterise the quasiplatonic surfaces as the local maximal for $|\operatorname{Aut} X|$, in the sense that, within the Teichmüller space of all compact Riemann surfaces of genus g, every other surface sufficiently close to X has fewer automorphisms.

5.2 Hurwitz Groups and Surfaces

Here we look for global maxima of $|\operatorname{Aut} X|$.

Problem: Given $g > 1$, what are the most symmetric Riemann surfaces of genus g?

We have $\operatorname{Aut} X \cong N(K)/K$, with $N(K)$ Fuchsian. The index $|N(K) : K|$ is finite, equal to the ratio of the areas of the fundamental regions of these two groups. For K this is $4\pi(g-1)$, so maximising $|\operatorname{Aut} X|$ is equivalent to minimising the area for $N(K)$. One can show that among all Fuchsian groups, this area is minimised by the triangle group $\Delta(3, 2, 7) = \Delta(2, 3, 7)$, given by

$$\Delta = \langle X, Y, Z \mid X^3 = Y^2 = Z^7 = XYZ = 1 \rangle.$$

Exercise 5.2. *Prove that Δ has a fundamental region of area $\frac{\pi}{21}$, and this is the minimum among all triangle groups. Use the Gauss-Bonnet formula: "area"$=\pi - \alpha - \beta - \gamma$ for a hyperbolic triangle with internal angles α, β, γ.*

This gives us the Hurwitz bound

$$|\operatorname{Aut} X| \leq \frac{4\pi(g-1)}{\pi/21} = 84(g-1),$$

attained iff $X \cong K\backslash\mathbb{H}$ where K is a normal subgroup of finite index in $\Delta = \Delta(3,2,7)$. (Every proper normal subgroup of finite index in Δ is torsion-free, easy exercise.) These surfaces X and finite groups $G = \operatorname{Aut} X$ are called *Hurwitz surfaces* and *Hurwitz groups*. These surfaces are all quasiplatonic.

Example 5.2. The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \Delta(3,2,\infty)$ maps onto $G = \mathrm{PSL}_2(7) = \mathrm{PSL}_2(\mathbb{Z}_7)$ by reducing coefficients $\mod 7$. The generator $Z : \tau \mapsto \tau + 1$ is mapped to an element z of order 7 in G, so G is a quotient Δ/K of $\Delta = \Delta(3,2,7)$

$$|G| = 168 \left(= \frac{7(7^2-1)}{2} \right),$$

so the surface $X = K\backslash\mathbb{H}$ has genus $g = 1 + \frac{168}{84} = 3$. This is Klein's quartic curve, given in projective coordinates by

$$x^3y + y^3z + z^3x = 0,$$

with $\operatorname{Aut} X \cong \mathrm{PSL}_2(7)$.

Exercise 5.3. *Prove that there is no Hurwitz group of genus 2.*

5.3 Kernels and Epimorphisms

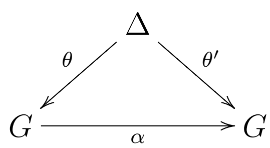
It's useful to count normal subgroups K of a triangle group Δ with a given quotient group $G \cong \Delta/K$.

Proposition 5.2. If Δ is any finitely generated group, and G is any finite group, the number $n_\Delta(G)$ of $K \trianglelefteq \Delta$ with $\Delta/K \cong G$ is given by

$$n_\Delta(G) = \frac{|\operatorname{Epi}(\Delta, G)}{|\operatorname{Aut} G|},$$

where $\operatorname{Epi}(\Delta, G)$ is the set of all epimorphisms $\theta : \Delta \to G$.

Proof. These normal subgroups K are the kernels of the epimorphisms $\theta : \Delta \to G$, and $\ker\theta = \ker\theta'$ iff $\theta' = \alpha \circ \theta$ for some $\alpha \in \operatorname{Aut} G$. Hence the kernels correspond to the orbits of $\operatorname{Aut} G$ acting by composition on $\operatorname{Epi}(\Delta, G)$.



Since $\operatorname{Aut} G$ acts semiregularly (i.e. $\alpha \circ \theta = \theta \Rightarrow \alpha = \mathrm{id}$), its orbits all have size $|\operatorname{Aut} G|$. By the hypotheses, $\mathrm{Epi}(\Delta, G)$ is finite, so the result follows. □

For many G, $|\operatorname{Aut} G|$ is known or easily found, so concentrate on counting epimorphisms. If Δ is a triangle group

$$\Delta(l, m, n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle,$$

finding epimorphisms $\Delta \to G$ is equivalent to finding triples $x, y, z \in G$ such that

a)
$$x^l = y^m = z^n = xyz = 1$$
(so there is a homomorphism $\Delta \to G : X \mapsto x$ etc.)

b) G is generated by x, y and z (or by any two of these), so we have an epimorphism.

If we want K to be torsion-free, we also require:

c) x, y and z must have orders exactly l, m and n.

5.4 Direct Counting

Example 5.3. Let $\Delta = \Delta(5, 2, \infty)$ and $G = A_5$, so we count $K \trianglelefteq \Delta$ with $\Delta/K \cong A_5$. This is equivalent to counting regular maps $\mathcal{M}$ ($m = 2$) with valency 5 ($l = 5$) and $\operatorname{Aut} \mathcal{M} \cong A_5$. A_5 has 24 elements x of order 5 (the 5-cycles), and 15 elements of order 2 (the double transpositions $(ab)(cd)$) giving $24 \times 15 = 360$ pairs x, y satisfying the relations of Δ. The subgroup $H = \langle x, y \rangle$ has order divisible by 10, so $H \cong D_5$ or $H = A_5$. There are 6 subgroups $H \cong D_5$, each generated by $4 \times 5 = 20$ pairs x, y, so 120 pairs don't generate A_5. Hence $360 - 120 = 240$ do generate A_5. Thus $|\mathrm{Epi}(\Delta, G)| = 240$. Now $\operatorname{Aut} A_5 = S_5$ (acting by conjugation) of order 120, so $n_\Delta(G) = \frac{240}{120} = 2$. Thus Δ has two normal subgroups K with $\Delta/K \cong A_5$, i.e. there are two regular 5-valent maps $\mathcal{M}$ with $\operatorname{Aut} \mathcal{M} \cong A_5$. One is the icosahedron, represented by

$$\theta : X \mapsto x = (1, 2, 3, 4, 5),\ Y \mapsto y = (1, 2)(3, 4),\ Z \mapsto z = (2, 5, 4).$$

The other one is the great dodecahedron, with 12 pentagonal faces, and the vertices and edges of an icosahedron. It's represented by

$$\theta : X \mapsto x = (1, 2, 3, 4, 5),\ Y \mapsto y = (1, 3)(2, 4),\ Z \mapsto z = (1, 2, 3, 5, 4).$$

This has genus $g = 1 + \frac{N}{2}(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}) = 4$ (where in this case $N = 60$ and $l = 5$, $m = 2$, $n = 3$.) The underlying algebraic curve is Bring's curve, given in $\mathbb{P}^4(\mathbb{C})$ by

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k = 0 \qquad (k = 1, 2, 3).$$

$\operatorname{Aut} X \cong S_5$ (permuting the coordinates), and the subgroup A_5 is the automorphism group of the map.

5.5 Counting by Character Theory

A (complex) *representation* of a group G is a homomorphism $\rho : G \to \mathrm{GL}(V)$, V a vector space over $\mathbb{C}$. $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ are *equivalent* if some isomorphism $V \to V'$ commutes with G. The representation ρ is *irreducible* is V has no G-invariant subgroups other than 0 and V. A finite group G has c irreducible representations, up to isomorphism, where c is the number of conjugacy classes in G. The *character table* of G is a $c \times c$ array, "entries" = "trace of $\rho(g)$ on each conjugacy class" (constant on each class).

Proposition 5.3. If $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are conjugacy classes in a finite group G, then the number of solutions of $xyz = 1$ in G with $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $z \in \mathcal{Z}$ is equal to

$$\frac{|\mathcal{X}| \cdot |\mathcal{Y}| \cdot |\mathcal{Z}|}{|G|} \cdot \sum_{\chi} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)},$$

where χ ranges over the irreducible characters of G.

In A_5 there are $c = 5$ conjugacy classes: the identity, 15 double transpositions, 20 3-cycles, and two classes of 12 5-cycles. Hence there are 5 irreducible characters and the character table looks like in table 1 where $\lambda, \mu = \frac{1 \pm \sqrt{5}}{2}$.

1	(..)(..)	(...)	$(.....)^+$	$(.....)^-$
1	1	1	1	1
3	1	0	λ	μ
3	1	0	μ	λ
4	0	1	-1	-1
5	1	-1	0	0

Table 1: The character table of A_5.

Example 5.4. Take $\Delta = \Delta(3,3,5)$, $G = A_5$ and count $K \trianglelefteq \Delta$ with $\Delta/K \cong A_5$. There is only one choice for the classes $\mathcal{X}$ and $\mathcal{Y}$ of elements x, y of order 3, and there are two choices for the class $\mathcal{Z}$ containing z of order 5. In each case there are 60 triples x, y, z in these classes satisfying $xyz = 1$, giving 120 triples. Hence there is $\frac{120}{120} = 1$ normal subgroup K. This gives a single regular bipartite map of type $(3,3,5)$ with $\mathrm{Aut}\,\mathcal{B} \cong A_5$. Exercise 2.3 $\Rightarrow$ genus $g = 5$. It's a double covering of the dodecahedron branched over the 12 face-centres, with vertices coloured alternately black and white.