Some Hints Concerning the Exercises

Exercise 1.1

F_n projective Fermat curve of exponent n, then $\mathrm{Aut}\, F_n \subseteq (C_n \times C_n) \rtimes S_3$ (semidirect product), see Gareth's Lecture 8. Moreover "=" holds if $n > 3$ because its surface group K is the commutator subgroup $[\Delta, \Delta]$ of the triangle group $\Delta = \langle n, n, n \rangle$, and $\mathrm{Aut}\, F_n = N(K)/K$, $N(K) = \langle 2, 3, 2n \rangle$ containing Δ as a normal subgroup with quotient S_3. In fact, the hyperbolic triangle for the construction of $\langle 2, 3, 2n \rangle$ results from that for $\langle n, n, n \rangle$ by barycentric subdivision, and it can be shown that $\langle 2, 3, 2n \rangle$ is maximal Fuchsian group, so $N(K)$ cannot be larger than $\langle 2, 3, 2n \rangle$ (the analogous statement for $n = 3$, i.e. for $\langle 2, 3, 6 \rangle$ would be definitely wrong!).

Exercise 1.2/4.2

The genus of the (compact) hyperelliptic curve with affine equation $y^2 = q(x)$, $\deg q = \left\{ \begin{matrix} 2g+1 \\ 2g+2 \end{matrix} \right\}$ with $\left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}$ points above $x = \infty$ is g by application of Riemann-Hurwitz to the mapping $f : (x, y) \mapsto x$ of degree 2: It is ramified in $2g + 2$ points with multiplicity 2, hence we have in fact

$$2g - 2 = 2 \cdot (-2) + \sum (\mathrm{mult}_p\, f - 1) = -4 + (2g + 2) \cdot 1.$$

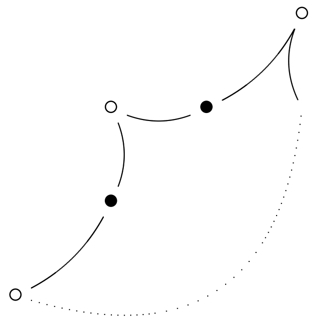
In the special case $q(x) = x^n - 1 = \prod_{k=1}^{n} (x - \zeta_n^k) \Rightarrow$

$$\left. \begin{matrix} X \to & \hat{\mathbb{C}} \to & \hat{\mathbb{C}} \\ (x, y) \mapsto & x \mapsto & x^n \end{matrix} \right\} \text{ ramified above } \begin{cases} 0 \text{ in 2 points, mult} = n \\ 1 \text{ in } n \text{ points, mult} = 2 \\ \infty \text{ in 1 point, mult} = 2n \text{ for } 2 | n \\ \infty \text{ in 2 points, mult} = n \text{ for } 2 \nmid n \end{cases}$$

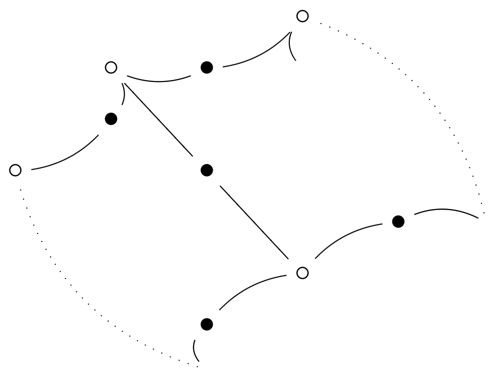
defines a Belyi function of $\deg 2n$ and the dessin for this Belyi function looks like two planes with this bipartite graph,

glued in the black points (and the point ∞ if n is even).

A picture in $\mathbb{H} \cong \mathbb{D}$ can be given and by a $2n$-sided polygon



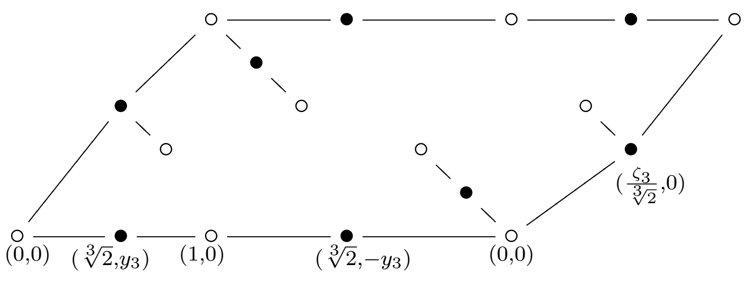
in the case $2|n$ and a $2(n-1)$-sided polygon, subdivided in two cells



if $2 \nmid n$.

Exercise 4.3

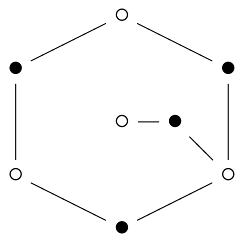
A Belyi function for $y^2 = x(x-1)(x - \frac{\zeta_3}{\sqrt[3]{2}}$ is formally the same $(x, y) \mapsto 4x^3(1-x^3)$ as if ζ_3 is replaced by 1, but with different ramifications, so the dessin now looks like



in a fundamental parallellogram for the elliptic curve. For $\zeta_3^2 = \bar{\zeta}_3$ instead of ζ_3 a mirror image of this dessin arises.

<u>Exercise 7.1</u>

$\beta \mapsto 1-\beta$ exchanges the colours of the bipartite graph, $\beta \mapsto \frac{1}{\beta}$ preserves $\beta = 1$ and exchanges zeros and poles, hence cell centers and 0-vertices $\Rightarrow$ the pole orders of β are the zero orders of that modified dessin $= \frac{1}{2}$ #"border edges of the cell", but the "inner edges" having the face on both sides have to be counted twice!



$$\beta \mapsto 16\beta(\beta - \frac{3}{4})^2$$

replaces $\circ_{0}\text{——}{}_{1}\bullet$ by $\circ_{0}\text{——}{}_{\frac{1}{4}}\bullet\text{——}{}_{\frac{3}{4}}\circ\text{——}{}_{1}\bullet$. (Idea: Take $\frac{1}{2}(1+T_n(2\beta-1))$, n odd, to insert more vertices with T_n the n^{th} Tshebychev polynomial)