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On the Moduli Space of Cyclic Trigonal Riemann Surfaces of Genus 4

Daniel Ying

Akademisk avhandling

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Finally, by using the same technique as in the case of cyclic trigonal Riemann surfaces of genus 4, we are able to deal with p-gonal Riemann surfaces and show that Accola's bound is sharp for p-gonal Riemann surfaces. Furthermore, we study families of p-gonal Riemann surfaces of genus $(p-1)^2$ with two p-gonal morphisms, and describe the structure of their moduli space.

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Thank you.

Daniel Ying

Linköping
2006-11-02

Contents

Introduction

Riemann surfaces have an appealing feature to mathematicians (and hopefully to non-mathematicians as well) in that they appear in a variety of mathematical fields.

The point of the introduction of Riemann surfaces made by Riemann, Klein and Weyl (1851-1913), was that Riemann surfaces can be considered as both one-dimensional complex manifolds and as smooth algebraic curves. This is called the Riemann functor and with it, the study of smooth algebraic curves is identical to the study of compact Riemann surfaces.

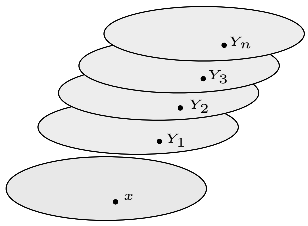
$$\left\{ \begin{array}{c} \text{Study of smooth} \\ \text{algebraic curves } C \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Study of compact} \\ \text{Riemann surfaces } X \end{array} \right\}$$

Another possibility is to study Riemann surfaces as two-dimensional real manifolds, as Gauss (1822) had taken on the problem of taking a piece of a smooth oriented surface in Euclidean space and embedding it conformally into the complex plane.

A fourth perspective came from the uniformization theory of Klein, Poincaré and Koebe (1882-1907), who showed that every Riemann surface (which by definition is a connected surface equipped with a complex analytic structure) also admits a Riemann metric.

Riemann surfaces were first introduced by Bernard Riemann in his doctoral dissertation *Foundations for a general theory of functions of a complex variable* in 1851. The use of the Riemann surfaces was as a topological aid to the understanding of many-valued functions.

Riemann's idea was to represent a relation (an algebraic curve) $P(x, y) = 0$ between x and y (complex variables) by covering a plane (or a sphere), representing the x variable, by a surface representing the y variable. Thus, the points on the y surface over a given point $x = \alpha$ were those values of y that satisfy the relation $P(x, y) = 0$. Locally this would look like the picture on the right, where $y_1, y_2, \ldots, y_n$ are the roots of the equation $P(\alpha, y) = 0$.



The sheets above x

Since then, the results have been improved, amongst others by F. Klein, but it took until 1913 for the first abstract definition of a Riemann surface to appear in H. Weyl's book *The concept of a Riemann surface.*

> A Riemann surface X is a Hausdorff connected topological space, together with a family $\{(\phi_j, U_j) : j \in J\}$ where U_j is an open cover of X and each ϕ_j is an homeomorphism of U_j onto an open subset of the complex plane.

Since then, the theory developed in lots of directions. One direction was the uniformization problem which was solved due to the work of Poincaré, Klein and Koebe. It was with this work that Poincaré (1882) discovered that the linear fractional transformations[1]

$$z \mapsto \frac{az + b}{cz + d}$$

give a natural interpretation of non-Euclidean geometry.

The underlying geometry of Riemann surfaces

The three main geometries (for surfaces) are given by the Euclidean geometry, the spherical geometry and the hyperbolic geometry, with curvature 0, 1 and -1 respectively, each geometry is connected to the Euclidean plane, the sphere and the hyperbolic plane. As models for these spaces we take the complex plane $\mathbb{C}$, the Riemann sphere $\widehat{\mathbb{C}}$ and the upper half-plane $\mathbb{H}$.
Working with the non-Euclidean geometry, one soon realizes that these transformations are groups of motions on the surfaces and Klein, who restated a result of Möbius (known as the Erlangen program), wanted to associate each geometry with a group of transformations that preserve its characteristic property.
Reformulating the geometry in this way, makes certain geometrical questions into questions about groups. One example of this is that regular

[1] Today known as Möbius transformations.

tessellations of a surface correspond to a subgroup of the full group of motions, consisting of those motions that map the tessellation onto itself. In the case of hyperbolic geometry, the interplay between geometric and group theoretic ideas was found to be very fruitful. Much of the work of Poincaré and Klein is built on these geometric, topological and combinatorial ideas. The main theorem of uniformization completely classifies all simply connected Riemann surfaces and from this it follows that every Riemann surface X is homeomorphic to some quotient space Y/G where Y is either $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}$, and G is a torsion free group acting discontinuously on Y.
In the case where X is a Riemann surface of genus $g \geq 2$ then X has underlying hyperbolic geometry. The corresponding group G in this case is a Fuchsian group, a discontinuous subgroup of the automorphism group of the hyperbolic plane.

Trigonal Riemann surfaces

Trigonal Riemann surfaces are generalizations of hyperelliptic surfaces. Hyperelliptic surfaces can be described by algebraic curves having equation

$$y^2 = p(x).$$

In this way they can be viewed as double coverings of the Riemann sphere. Furthermore, hyperelliptic Riemann surfaces have a unique hyperelliptic involution (2-gonal morphism).
Naturally, one can ask about what happens when we generalize to p-gonal Riemann surfaces. Does p-gonal Riemann surfaces still have a unique p-gonal morphism?
Accola showed in his paper [1] that p-gonal Riemann surfaces of genus $g \geq (p-1)^2$ the p-gonal morphism is unique.
A closed Riemann surface X which can be realized as a 3-sheeted (branched) covering of the Riemann sphere is said to be *trigonal*, and such a covering is called a *trigonal morphism*. A morphism is a branched covering.
Since the study of Riemann surfaces is equivalent to the study of algebraic curves, a trigonal Riemann surface X is represented by an algebraic curve of the form

$$y^3 + yb(x) + c(x) = 0$$

If $b(x) \equiv 0$ then the trigonal morphism is a *cyclic regular covering* and the Riemann surface is called *cyclic trigonal*. A non-cyclic trigonal Riemann surface is said to be a *generic trigonal Riemann surface.*
If X_g is a cyclic trigonal Riemann surface then there exists an automorphism, φ, of order 3 such that $X_g/\langle\varphi\rangle$ is the Riemann sphere with conic points of order 3, and in fact there are $g+2$ such conical points. φ will be called a trigonal morphism. Cyclic Riemann surfaces have equations

$$y^3 = c(x)$$

A trigonal Riemann surface X_g, of genus g, can be uniformized by a Fuchsian group Γ, that is, $X_g = \mathbb{H}/\Gamma$ and the trigonal morphism gives $g+2$ conical points on the Riemann sphere so the quotient surface is uniformized by a Fuchsian group with signature $s(\Lambda) = (0;3,3,\ldots,3,3)$.
Let $G = Aut(X_g)$, then the quotient surface X_g/G is also uniformized by a Fuchsian group Δ such that $X_g/G = \mathbb{H}/\Delta$.

$$\begin{tikzcd} X_g \cong \mathbb{H}/\Gamma \arrow[dd] \arrow[dr] & \\ & \begin{array}{l} \mathbb{H}/\Lambda = X_g/\langle\varphi\rangle = \\ = \widehat{\mathbb{C}} \text{ with conical points of order } 3 \\ s(\Lambda) = (0;3,3,\ldots,3,3) \end{array} \arrow[dl] \\ X_g/G \cong \mathbb{H}/\Delta & \end{tikzcd}$$

In Accola's paper [1], the trigonal morphism, φ, is unique for Riemann surfaces of genus $g \geq 5$, however for genus less than 5 the morphism need not be unique as we shall see.
The task of classifying the possible trigonal Riemann surfaces of genus 4 is done by finding the signature groups of Δ and defining suitable epimorphisms from the groups Δ onto groups with order a multiple of 3.
Having shown that cyclic trigonal Riemann surfaces of genus 4 exist, the next is to study the space of such surfaces.
The moduli space of Riemann surfaces is the space obtained by defining two Riemann surfaces to be equivalent if they have the same conformal structure.
As it turns out, the moduli space of cyclic trigonal Riemann surfaces of genus 4 is a disconnected space of complex dimension 3 and the surfaces generate subspaces (equisymmetric strata) inside this space.
An equisymmetric strata is a subspace of the moduli space, corresponding to surfaces having the same symmetry type. This means that two surfaces are equisymmetric if their conformal automorphism groups determine conjugate subgroups of the mapping class group (moduli group).
As a consequence of the result, we are able to show that in the case of genus 4 Accola's bound is sharp. Furthermore, extending the result to cyclic p-gonal Riemann surfaces we can again state that Accola's bound is sharp for general prime p.

Outline of the thesis

Chapter 1: Preliminaries

Section 1.1 This section deals with the hyperbolic geometry which is the underlying geometry of the Riemann surfaces of genus greater than one.

This will become important when defining the elements of the Fuchsian groups in section 1.3.
Section 1.2 In here, the main theory of Riemann surfaces is presented and the analytic structure of the Riemann surfaces is defined.
Section 1.3 The section on Uniformization is rather large and deals with the group theoretical part of Riemann surfaces and smooth and branched coverings of Riemann surfaces. The well-known results of uniformization of Riemann surfaces is explained and the Fuchsian groups uniformizing the Riemann surfaces are presented here.
Section 1.4 Teichmüller theory is the underlying theory for classifying the space of Riemann surfaces. Here the structure of the moduli space is developed.
Section 1.5 This section gives the background of the theory of equisymmetric Riemann surfaces. It involves finite group actions on Riemann surfaces and the description of the algebraic structure of such groups.

Chapter 2: Cyclic trigonal Riemann surfaces of genus 4

This chapter is the main part of the thesis. It involves the calculations of cyclic trigonal Riemann surfaces of genus 4.
Section 2.1 This section states the definition of trigonal Riemann surfaces and mentions some results on trigonal Riemann surfaces.
Section 2.2 This section shows the existence of cyclic trigonal Riemann surfaces of genus 4, and that there are cyclic trigonal Riemann surfaces of genus 4 with non-unique trigonal morphisms.
Section 2.3 Shows that there is a uniparametric family of cyclic trigonal Riemann surfaces admitting several cyclic trigonal morphisms. The family consists of Riemann surfaces of genus 4 with automorphism group $D_3 \times D_3$ and there is also one Riemann surface Y_4 in the family, with automorphism group $(C_3 \times C_3) \rtimes D_4$.
Section 2.4 Describes the space of cyclic trigonal Riemann surfaces, and shows that this forms a disconnected subspace of the moduli space $\mathcal{M}_4$ of Riemann surfaces of genus 4. This subspace has complex dimension three.
Section 2.5 This section describes the relation between the stratas of trigonal Riemann surfaces from section 2.4. Furthermore, we classify the spaces of cyclic trigonal Riemann surfaces of genus 4 found in section 2.2 and show which subspace of $\mathcal{M}_4$ they belong to.

Chapter 3: Cyclic p-gonal Riemann surfaces

This chapter is devoted to the extension of the result to cyclic p-gonal Riemann surfaces and shows that there is a family of cyclic p-gonal Riemann surfaces admitting several p-gonal morphisms.
Section 3.1 Starts off by defining p-gonal Riemann surfaces and characterizing them using Fuchsian groups. Then we show the existence of cyclic

p-gonal Riemann surfaces with several trigonal morphism, using a similar algorithm as in the case of cyclic trigonal Riemann surfaces of genus 4. **Section 3.2** Shows the existence of a unique class of actions on the surfaces, showing that the spaces of cyclic p-gonal Riemann surfaces ${}^{4p^2}\mathcal{M}^{p}_{(p-1)^2}$, admitting several trigonal morphisms, are Riemann surfaces and each of them being identifiable with the Riemann sphere with tree punctures.

Chapter 4: Conclusions and further work

We state some of the open questions.

Appendix A: List of groups

This is a list of the important groups that are used in this work.

Chapter 1

Preliminaries

The surface was invented by the devil.

-Wolfgang Pauli

1.1 Hyperbolic geometry

The connection between Fuchsian groups and hyperbolic geometry was made by Henri Poincaré and published in 1882. Fuchsian groups arise when studying compact Riemann surfaces and we will see more about this in section 1.3. For now we will turn our attention toward the geometry of the hyperbolic plane, in which all the actions will take place.
We like to describe the hyperbolic geometry in terms of Euclidean geometry and this is done by using different models. The two most common such models are the upper complex half-plane,

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : \quad y > 0\}$$

and the unit disc,

$$\mathbb{D} = \{z \in \mathbb{C} : \quad |z| < 1\}$$

In order for these to be topological spaces we assign a metric ρ in each case given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad z = x + iy \in \mathbb{H}$$

for the upper half-plane and

$$ds = \frac{2|dz|}{1 - |z|^2}, \quad z \in \mathbb{D}$$

for the unit disc.

There is a natural map between these two models given by

$$z \mapsto \frac{z-i}{z+i}$$

sending i to the origin, 0 to -1 and the point at infinity to 1. Due to the analyticity of this map (outside certain points of course) we can choose to work on either model without having to specify the theory to that particular model. Hence, we will make no distinction between these two models and the context makes clear which of them is being used.
The group of orientation preserving isometries (that is, maps leaving the metric ρ invariant) of the hyperbolic plane $\mathbb{H}$ is the group of Möbius transformations, $PSL_2(\mathbb{R})$[1]. Elements of this group are expressions of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Note that the group $PSL_2(\mathbb{R})$ can be represented as a group of equivalence classes of 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and so it will make sense to classifying these isometries using matrix operation such as trace of a matrix.
The extended group of Möbius transformation also contains the orientation reversing isometries given by

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$$

such that $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$.

Classifying isometries of $\mathbb{H}$

We define a hyperbolic line (h-line) to be the intersection of the hyperbolic plane with a Euclidean circle or straight line, which is orthogonal to the circle at infinity. Using this definition it is a well know fact that the following hold:

1. The reflection in an h-line is a ρ-isometry.

2. Any transformation is the product of at most three reflections. Furthermore, the orientation preserving isometries of $\mathbb{H}$ are products of exactly two reflections.

3. If L is an h-line and g is an hyperbolic isometry then $g(L)$ is an h-line.

[1]PSL stands for the projective special linear group

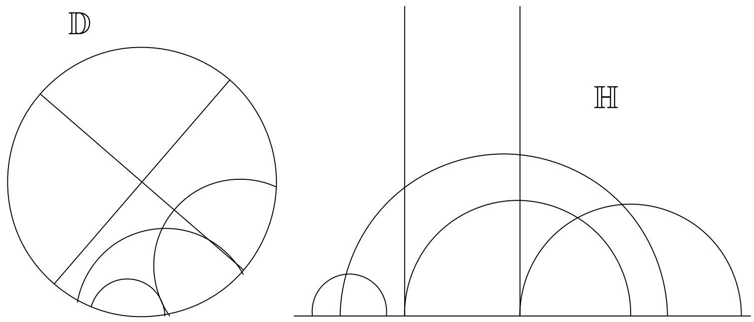


Figure 1.1: Hyperbolic lines in the unit disk and in the upper half-plane

4. Given any two h-lines L_1 and L_2, there is a ρ-isometry g such that $g(L_1) = L_2$.

A conformal (orientation preserving) isometry is of one of the three types: parabolic, elliptic or hyperbolic. The type of isometry can be recognized by the location of the fixed points or by the trace of the corresponding matrix.

1. **Parabolic isometries:** An isometry g is parabolic if and only it can be represented as $g = \sigma_1\sigma_2$ where σ_j is a reflection in the geodesic L_j and L_1 and L_2 are parallel geodesics. Using the trace this becomes $Trace^2(g) = 4$.

2. **Elliptic isometries:** An isometry g is elliptic if and only if it can be represented as $g = \sigma_1\sigma_2$ where σ_j is the reflection in L_j and L_1 and L_2 intersect at a point w. $Trace^2(g) \in [0, 4)$

3. **Hyperbolic isometries:** An isometry g is hyperbolic if and only if it can be represented as $g = \sigma_1\sigma_2$ where σ_j is a reflection in the geodesic L_j and L_1 and L_2 are disjoint and have L_0 as the common orthogonal geodesic. $Trace^2(g) \in (4, +\infty)$

We can calculate the hyperbolic area of triangles using the Gauss-Bonnet formula, and by generalizing this, we can also get the area of general polygons in the hyperbolic plane. The area $\mu(T)$ of a hyperbolic triangle T with angles $\alpha_1, \alpha_2, \alpha_3$ is given by

$$\mu(T) = \pi - (\alpha_1 + \alpha_2 + \alpha_3).$$

The area of a hyperbolic polygon P_n with angles $\alpha_1, \ldots, \alpha_n$ is given by

$$\mu(P_n) = (n-2)\pi - (\alpha_1 + \ldots + \alpha_n)$$

1.2 Riemann surfaces

The following section is a survey of the theory of Riemann surfaces. This material can be found [2], [25], [26], [32] and [34].

Locally a Riemann surface has the structure of the complex plane. In other words it can be viewed as a real 2 dimensional manifold. The difference between them is that Riemann surfaces can have singular points, whereas a manifold is smooth at every point. We will start by defining this notion in a rigorous way and from there we will develop some of the theory of Riemann surfaces.
Let X be a topological space. In order to make X look like the complex plane we will define local coordinates on X such that we will be able do the usual calculations such as calculating function values of points on the Riemann surface and do integration on the surface. To do this we use complex charts on the surface, sending points homeomorphically to the complex plane. More strictly we have

Definition 1.2.1. A **complex chart** or simply a **chart** on a topological surface X is a homeomorphism $\phi : U \to V$, where $U \subset X$ is an open subset of X and $V \subset \mathbb{C}$ is an open subset of the complex plane. We say that a chart ϕ is centered at $p \in U$ if $\phi(p) = 0$.

Now, the notion of a chart on a Riemann surface is clearly not unique by the definition. Whenever two charts on a Riemann surface overlap we have the following.

Definition 1.2.2. Let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be two complex charts on a surface X. Then ϕ_1 and ϕ_2 are **compatible** if either $U_1 \cap U_2 = \emptyset$ or

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is a holomorphic[2] function.

The above definition allows us to switch between different local complex coordinates in a smooth way such that the function $h = \phi_2 \circ \phi_1^{-1}$ sending a local coordinate z to another local coordinate $w = h(z)$ is holomorphic.
Now any topological surface can be covered by a collection of open sets $\{U_\alpha\}$ and defining local coordinates on each such set we obtain the notion of a complex atlas.

Definition 1.2.3. A **complex atlas** $\mathcal{A}$ on X is a collection

$$\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$$

of pairwise compatible charts whose domains cover X, that is $X = \cup_\alpha U_\alpha$.

[2]A holomorphic function is simply an analytic function on some region R

Two complex atlases are equivalent if and only if their union is also a complex atlas. Moreover, every complex atlas is contained in a unique maximal complex atlas and two atlases are equivalent if and only if they are contained in the same maximal complex atlas.
We are now ready to define the notion of complex structure on a surface.

Definition 1.2.4. A **complex structure** on a surface X is a maximal complex atlas on X, or equivalently, an equivalence class of complex atlases on X.

Now that we have laid the foundations of the theory we are ready to introduce the concept of a Riemann surface.

Definition 1.2.5. A **Riemann surface** is a second countable connected Hausdorff topological space X together with a complex structure.

By the definition of the structure on the Riemann surface it is not difficult to see that every Riemann surface can be viewed as a 2-dimensional $\mathcal{C}^\infty$ real manifold (there may be singular points).
There are many examples of Riemann surfaces.

Example 1.2.6. The Complex plane. Clearly this can be made into a Riemann surface simply by using the identity map as the complex chart.

Example 1.2.7. The Riemann Sphere. Usually this is denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which is simply the complex plane together with a point of infinity. This is the same as adding a line at infinity which also gives us that the Riemann sphere can be described as the projective line $\mathbb{P}^1(\mathbb{C})$. From real geometry the Riemann sphere is simply the 2-sphere embedded in 3 dimensions.
One possible atlas on $\widehat{\mathbb{C}}$ is to take $\{(\varphi_i, U_i)\}$ for $i = 1, 2$ to be $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}\backslash\{0\} \cup \{\infty\}$. The homeomorphisms in this case can be defined to be $\varphi_1 = 1_d$ and $\varphi_2 = 1/z$ for $z \in \mathbb{C}$ and $\varphi_2 = 0$ for $z = \infty$.
Clearly, $\widehat{\mathbb{C}} = U_1 \cup U_2$ by definition, and φ_1, φ_2 are homeomorphisms. Their compositions $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$ and $\varphi_1 \circ \varphi_2^{-1}(z) = 1/z$ are holomorphic on $\varphi_1(U_1 \cap U_2) = \mathbb{C}\backslash\{0\}$ and $\varphi_2(U_1 \cap U_2) = \mathbb{C}\backslash\{0\}$ respectively. Hence, the atlas is holomorphic and gives the desired complex structure on $\widehat{\mathbb{C}}$.

Example 1.2.8. The affine Fermat curve, $F_n^{aff} = \{(x, y) \in \mathbb{C}^2 | x^n + y^n = 1\}$. Affine algebraic curves can be represented as Riemann surfaces (and vice versa).
Take as charts for example $(x, y) \mapsto y$, which is holomorphic in suitable neighbourhoods of all points except when $x = 0$, $y^n = 1$, and similarly $(x, y) \mapsto x$ which is holomorphic in suitable neighbourhoods of all points except for $y = 0$, $x^n = 1$. The transition functions are given by $x = \sqrt[n]{1 - y^n}$ and $y = \sqrt[n]{1 - x^n}$.

Holomorphic functions on a Riemann surface

Using the complex structures on the Riemann surfaces we are able to define functions acting on the Riemann surfaces via the complex charts. In the following, let X be a Riemann surface, p a point of X and let $f : X \to \mathbb{C}$ be a complex valued function defined in a neighbourhood W of p.

Definition 1.2.9. The function f is **holomorphic** at p if there exists a chart $\phi : U \to V$ with $p \in U$ such that the composition $f \circ \phi^{-1} : \mathbb{C} \to \mathbb{C}$ is a holomorphic function (in the usual sense) at $\phi(p) \in \mathbb{C}$. f is holomorphic in W if it is holomorphic at every point of W.

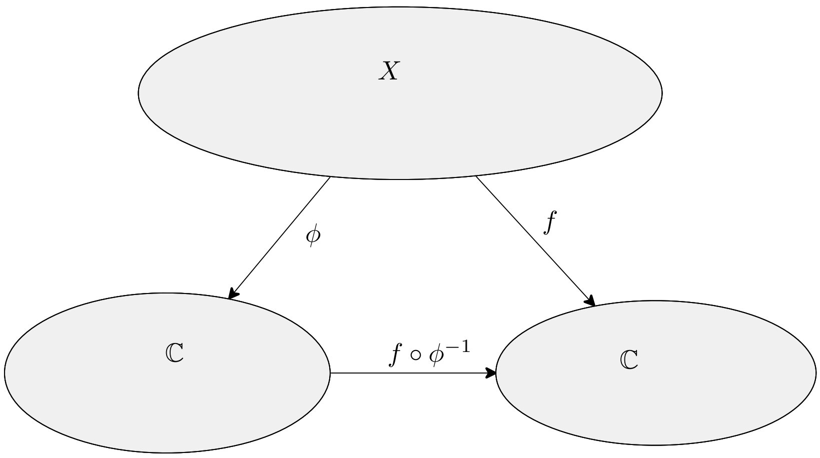


Figure 1.2: Holomorphic maps between Riemann surfaces

The set of holomorphic functions defined on an open set $W \subset X$ on a Riemann surface X is denoted by

$$\mathcal{O}_X(W) = \mathcal{O}(W) = \{f : W \to \mathbb{C} \quad | \quad f \quad \text{is holomorphic}\}$$

As with normal functions there are special points in the domain that needs to be carefully explored.

Definition 1.2.10. Let f be holomorphic in a punctured neighbourhood of $p \in X$. Then

1. f has a **removable singularity at** p if and only if there exists a chart $\phi : U \to V$ with $p \in U$ such that the composition $f \circ \phi^{-1}$ has a removable singularity at $\phi(p)$.

2. f has a **pole at** p if and only if there exists a chart $\phi : U \to V$ with $p \in U$ such that the composition $f \circ \phi^{-1}$ has a pole at $\phi(p)$.

3. f has an **essential singularity at** p if and only if there exists a chart $\phi : U \to V$ with $p \in U$ such that the composition $f \circ \phi^{-1}$ has an essential singularity at $\phi(p)$.

There is a straight forward way of deciding which type of singularity a holomorphic function has, at a point p, by investigating the behavior of $f(x)$ for x near p:

1. If $f(x)$ is bounded in a neighbourhood of p, then f has a removable singularity at p. Moreover, when the limit $\lim_{x\to p} f(x)$ exists and is defined to be $f(p)$ then f is holomorphic at p.

2. If f approaches ∞ as $x \to p$ then f has a pole at p.

3. If $|f(x)|$ has no limit as $x \to p$ then f has an essential singularity at p.

Meromorphic functions on a Riemann surface

Even though the holomorphic functions have singularities there is a way to naturally move around these points by defining meromorphic functions.

Definition 1.2.11. A function f on X is **meromorphic** at a point $p \in X$ if it is either holomorphic, has a removable singularity, or has a pole at p. f is meromorphic on an open set W if it is meromorphic at every point of W.

The set of meromorphic functions defined on an open set $W \subset X$ on a Riemann surface X is denoted

$$\mathcal{M}_X(W) = \mathcal{M}(W) = \{f : W \to \mathbb{C} \quad | \quad f \quad \text{is meromorphic}\}$$

Laurent series

Let f be defined and holomorphic in a punctured neighbourhood of $p \in X$. Let $\phi : U \to V$ be a chart on X with $p \in U$ and let z be the local coordinate on X near p so that $z = \phi(x)$ for x near p. Then $f \circ \phi^{-1}$ is holomorphic in a neighbourhood of $z_0 = \phi(p)$. Therefore $f \circ \phi^{-1}$ can be expanded in a Laurent series about z_0:

$$f(\phi^{-1}(z)) = \sum_n c_n (z - z_0)^n$$

This is called the **Laurent series for** f **about** p with respect to ϕ. The coefficients $\{c_n\}$ are called the Laurent coefficients. Using the Laurent coefficients we can also decide the types of the singularities:

Lemma 1.2.12. *With the above notation we have:*

1. *f has a removable singularity at p if and only if any one of its Laurent series has no terms with negative powers.*
2. *f has a pole at p if and only if one of its Laurent series has finitely many nonzero terms with negative powers.*
3. *f has an essential singularity at p if and only if any one of its Laurent series has infinitely many terms with negative powers.*

The order of a meromorphic function

Since the meromorphic functions are expandable into Laurent series around the singular points, and by the behavior of the series at such points, it is natural to define the order of a meromorphic function at such points.

Definition 1.2.13. Let f be meromorphic at a point p with Laurent series in local coordinate z being $\sum_n c_n(z-z_0)^n$. The **order of f at p**, denoted $\mathrm{ord}_p(f)$, is defined to be the minimum exponent appearing in the Laurent series

$$\mathrm{ord}_p(f) = \min\{n | c_n \neq 0\}$$

Using the order of a meromorphic function, the classification of the function value at a point p is given by the following lemma:

Lemma 1.2.14. *Let f be meromorphic at p. Then*

1. *f is holomorphic at p if and only if $ord_p(f) \geq 0$.*
2. *$f(p) = 0$ if and only if $ord_p(f) > 0$.*
3. *f has a pole at p if and only if $ord_p < 0$.*
4. *f has neither a pole nor a zero at p if and only if $ord_p(f) = 0$.*

Definition 1.2.15. A function f is said to have a **zero of order** n at p if $\mathrm{ord}_p(f) = n \geq 1$. A function f is said to have a **pole of order** n at p if $\mathrm{ord}_p(f) = -n < 0$.

A well-known result about the meromorphic functions is the following theorem.

Theorem 1.2.16. *Any meromorphic function on the Riemann sphere is a rational function.*

That is any meromorphic function on the Riemann sphere can be written as $p(z)/q(z)$ where $p(z)$ and $q(z)$ are complex polynomials in $\mathbb{C}[z]$.

Corollary 1.2.17. *Let f be any meromorphic function on the Riemann sphere. Then*

$$\sum_{p \in \widehat{\mathbb{C}}} ord_p(f) = 0$$

Hence a meromorphic function on the Riemann sphere has the same number of poles and zeros.

Holomorphic maps between Riemann surfaces

Of course, by the definition of the complex charts on the Riemann surfaces and the definition of holomorphic maps acting on Riemann surfaces it is also possible to define holomorphic maps between Riemann surfaces in a natural way, using the inverse of one of the complex charts.

Definition 1.2.18. A mapping $F : X \to Y$ between two Riemann surfaces X and Y is **holomorphic at** $p \in X$ if and only if there exists charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$ and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$ such that the composition $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$.

If F is defined on an open subset $W \subset X$, then F is holomorphic on W if F is holomorphic at each point of W. In particular, F is a holomorphic map if and only if F is holomorphic on all of X.

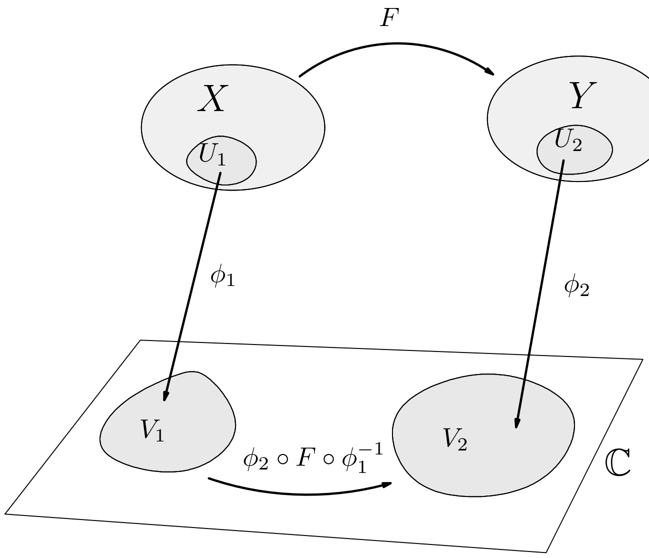


Figure 1.3: Holomorphic maps between Riemann surfaces

Lemma 1.2.19. *Let $F : X \to Y$ be a mapping between Riemann surfaces.*

1. *If F is holomorphic, then F is continuous and $\mathcal{C}^\infty$.*
2. *The composition of holomorphic maps is holomorphic.*
3. *The composition of a holomorphic map with a holomorphic function is holomorphic.*
4. *The composition of a holomorphic map with a meromorphic function is meromorphic. That is, if $F : X \to Y$ is a holomorphic map between Riemann surfaces and g is a meromorphic function on $W \subset Y$ then $g \circ F$ is a meromorphic function on $F^{-1}(W)$.*

Note that as a technical detail in the last statement, the image of F, $F(X)$, can not be a subset of the poles of g.

Also from the last property we are able to define the **pullback** of a holomorphic map between Riemann surfaces. Let $F : X \to Y$ be a holomorphic map, then for every set $W \subset Y$, F induces a $\mathbb{C}$-algebra homomorphism

$$F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W))$$

defined by composition with F, namely $F^*(g) = g \circ F$. Likewise, we can define the pullback for meromorphic functions

$$F^* : \mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}(W))$$

again defined by $F^*(g) = g \circ F$, for F nonconstant. Moreover, if $F : X \to Y$ and $G : Y \to Z$ are holomorphic maps, then

$$F^* \circ G^* = (G \circ F)^*$$

Isomorphisms and Automorphisms

Definition 1.2.20. An **isomorphism**[3] between Riemann surfaces is a holomorphic map $F : X \to Y$ which is bijective and whose inverse $F^{-1} : Y \to X$ is holomorphic. A self-isomorphism $F : X \to X$ is called an **automorphism** of X. If there exist an isomorphism between two Riemann surfaces X and Y, they are said to be isomorphic.

Properties of holomorphic maps

From the previous definitions, many properties of holomorphic maps follows. First is a result about the local form of holomorphic maps.

Proposition 1.2.21. *Let $F : X \to Y$ be a holomorphic map defined at $p \in X$, which is nonconstant. Then there is a unique integer $m \geq 1$ such that for every chart $\phi_2 : U_2 \to V_2$ on Y centered at $F(p)$ there exists a chart $\phi_1 : U_1 \to V_1$ on X, centered at p, such that*

$$\phi_2(F(\phi_1^{-1}(z))) = z^m$$

[3]Sometimes also called a biholomorphism.

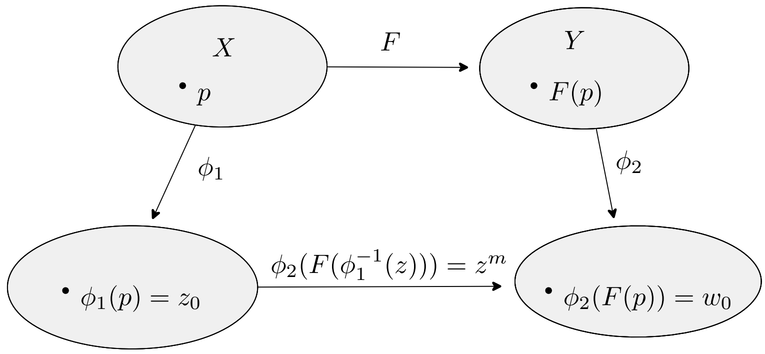


Figure 1.4: The local form of a holomorphic map, and the local coordinates in the $\mathbb{C}$-plane.

Definition 1.2.22. The **multiplicity of** F **at** p, denoted $\mathrm{mult}_p(F)$, is the unique integer m such that there are local coordinates near p and $F(p)$ with F having the form $z \mapsto z^m$.

Now, take any local coordinates z near p and w near $F(p)$, such that $\phi_1(p) = z_0$ and $\phi_2(F(p)) = w_0$. Then, in terms of these coordinates the map F can be written as

$$w = h(z)$$

for a holomorphic map h and then of course $h(z_0) = w_0$.

Lemma 1.2.23. *With the above notation, the multiplicity of F at p is one more than the order of vanishing of the derivative $h'(z_0)$ of h at z_0:*

$$mult_p(F) = 1 + ord_p(\frac{dh}{dz})$$

In particular, the multiplicity is the exponent of lowest strictly positive term of the power series for h, that is if

$$h(z) = h(z_0) + \sum_{i=m}^{\infty} c_i(z - z_0)^i$$

with $m \geq 1$ and $c_m \neq 0$, then $mult_p(F) = m$.

Definition 1.2.24. Let $F : X \to Y$ be a nonconstant holomorphic map. A point $p \in X$ is a **ramification point** for F if $\mathrm{mult}_p(F) \geq 2$. A point $y \in Y$ is a **branch point** for F if it is the image of a ramification point.

Example 1.2.25. An easy way of seeing the ramification is to look at a smooth complex algebraic curve (a compact Riemann surface). Let the curve $\mathcal{C}$ be given by

$$y^2 = x(x - 1)(x - \lambda)$$

This curve, as a covering of the Riemann sphere, is ramified at the points $0, 1, \lambda$ and ∞. All other points are unramified and have multiplicity 1.

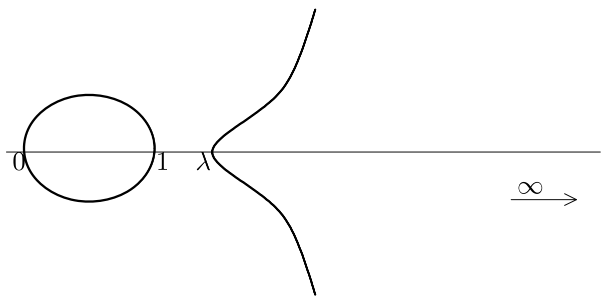


Figure 1.5: The curve $y^2 = x(x-1)(x-\lambda)$

Lemma 1.2.26. *Let f be a meromorphic function on a Riemann surface X with associated holomorphic map $F : X \to \widehat{\mathbb{C}}$.*

1. *If $p \in X$ is a zero of f, then*
$$mult_p(F) = ord_p(f)$$
2. *If p is a pole of f, then*
$$mult_p(F) = -ord_p(f)$$
3. *If p is neither a zero nor a pole of f, then*
$$mult_p(F) = ord_p(f - f(p))$$

The degree of a holomorphic map between Riemann surfaces

Proposition 1.2.27. *Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. For each $y \in Y$, define $d_y(F)$ to be the sum of multiplicities of F at the points of X mapping to y:*

$$d_y(F) = \sum_{p \in F^{-1}(y)} mult_p(F)$$

Then $d_y(F)$ is constant, independent of y.

From this global property of $d_y(F)$ the notion of degree of a holomorphic map is possible.

Definition 1.2.28. Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. **The degree of** F, denoted $deg(f)$, is the integer $d_y(F)$ for any $y \in Y$.

Corollary 1.2.29. *A holomorphic map between Riemann surfaces is an isomorphism if and only if it has degree one.*

The sum of the orders of a meromorphic function

There is now a possibility to generalize the statements of corollary 1.2.17.

Proposition 1.2.30. *Let f be a nonconstant meromorphic function on a compact Riemann surface X. Then*

$$\sum_{p \in X} ord_p(f) = 0$$

The Euler characteristic and the Riemann Hurwitz formula

The Euler characteristic (or Euler number) is a well known concept in graph theory and many other areas. Here it will be presented on smooth compact surfaces, or more specifically on compact 2-manifolds.
A triangulation of a surface is basically a grid of triangles on the surface. Clearly any given triangulation is not unique, however, there is an invariant of such triangulations given by the Euler characteristic.

The Euler number of a compact surface

Definition 1.2.31. Let S be a compact 2-manifold, possibly with boundary. Suppose a triangulation of S is given with v vertices, e edges, and f faces. **The Euler characteristic** (or sometimes **Euler number** of S) , denoted $\chi(S)$, with respect to the triangulation is the integer

$$\chi(S) = v - e + f$$

However, as Euler discovered there is a connection between the number of vertices, edges and faces (triangles) of each such triangulation:

Proposition 1.2.32. *The Euler characteristic is independent of the triangulation. Moreover, for a compact orientable 2-manifold S without boundary points of topological genus g, the Euler characteristic is*

$$\chi(S) = 2 - 2g$$

There is a connection between two Riemann surfaces and their genus known as the Riemann-Hurwitz' formula and this is given by the following theorem.

Theorem 1.2.33. *(Riemann-Hurwitz) Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces of genus g_X and g_Y respectively. Then*

$$2g_X - 2 = deg(F)(2g_Y - 2) + \sum_{p \in X} \Big(mult_p(F) - 1 \Big)$$

It is not difficult to see why this is true. If F is unramified, then all fibers of F have $\deg(F)$ points. Hence if we assume that we have a triangulation of the surface Y then each such triangle is lifted to $\deg(F)$ triangles in X. Now, in the case there is ramification, we have to adjust the formula. Assume that the triangulation of Y has vertices in each of the branch points. Then the pre-image of such triangulation, centered at the branch point p with say n edges leaving it, gives us a vertex with $n \cdot (\mathrm{mult}_p(F) - 1)$ edges leaving it.

1.3 Uniformization

The uniformization problem was first stated as a problem of parameterizing algebraical curves with a single parameter. The solution of the uniformization problem would depend on a better understanding of surfaces, and as such, this led to the study of the topology of the surfaces, the periodicities associated with their closed curves, and the way these periodicities could be reflected in $\mathbb{C}$. These problems where attacked by Poincaré and Klein in the 1880's and their work led to the positive solution of the uniformization problem by Poincaré and Koebe (1907).
However, the importance of this work does not lie in the solution of the uniformization problem, instead it lies in the preliminary work done by Poincaré and Klein. Their discovery, that multiple periodicities were reflected in $\mathbb{C}$ by groups of transformation, and that these transformations are of the types

$$z \mapsto \frac{az+b}{cz+d},$$

plays an important role in modern theory of Riemann surfaces.
Before we can state the results of this, we need to define some of the underlying group theory of Riemann surfaces.

Group actions on Riemann surfaces

Having laid the foundations of the analytic part of Riemann surfaces the focus will now turn to the more algebraic view of Riemann surfaces. Mostly this will involve finite groups and finitely generated groups and the following is a quick review of the group theory involved. This can be found in the books [2], [5], [25], [32], [34] and [36].

Finite group actions

Let G be a finite group and X a Riemann surface. We begin by defining the notion of a group acting on a Riemann surface:

Definition 1.3.1. The **action of G on X** is a map $G \times X \to X$ given by

$$(g, p) \mapsto g \cdot p$$

such that

1. $(gh) \cdot p = g \cdot (h \cdot p)$ for $g, h \in G$ and $p \in X$.
2. $e \cdot p = p$ for $p \in X$ and $e \in G$ where e is the identity.

For fixed $g \in G$ the map sending p to $g \cdot p$ is a bijection and clearly the inverse is given by $g^{-1} \cdot p$.

Definition 1.3.2. The orbit of $p \in X$ is the set of points

$$G(p) = \{g \cdot p \in X | g \in G\}.$$

The stabilizer of $p \in X$ is the subgroup

$$G_p = \{g \in G | g(p) = p\},$$

(sometimes written $stab(p)$).

It is easy to show the fact that points in the same orbit have the conjugate stabilizers, namely that

$$G_{g(p)} = g\ G_p\ g^{-1}.$$

Also, if G is a finite group, then its order is divisible by the order of the orbits and stabilizer group

$$|G(p)| \cdot |G_p| = |G|.$$

The **kernel** of an action of G on X is the subgroup

$$K = \{g \in G | g \cdot p = p \quad \forall p \in X\}.$$

The kernel K is the intersection of all stabilizer subgroups and is clearly a normal subgroup of G. The quotient group G/K acts on X with trivial kernel and identical orbits to the G action.
Thus, one may assume that the kernel is trivial (or else we just work with the quotient group). Such action is called an **effective** or **faithful action**.
The action is **continuous/holomorphic** if for every $g \in G$, the bijection sending p to $g \cdot p$ is a **continuous/holomorphic** map from X to itself. If the map is holomorphic it will necessarily be an automorphism of X.
If the stabilizer G_p is trivial for all p then G is said to **act freely** on X. This is equivalent to having a small neighbourhood U_p around each point $p \in X$ such that $g(U_p) \cap U_p = \varnothing$ for all $g \in G$

The quotient space X/G

The quotient space X/G is the set of orbits of G and there is a natural (continuous) map sending each point to its orbit

$$\pi : X \to X/G.$$

We give a topology to the orbit space X/G by declaring a subset $U \subset X/G$ to be open if and only if $\pi^{-1}(U)$ is open in X. This is simply the quotient topology on X/G.
Now that the quotient space is endowed with a topology we need to show that it is a Riemann surface, i.e. that we can put a complex structure on it such that the quotient map π is a holomorphic map.
First of all we need some facts about the stabilizer subgroups.

Proposition 1.3.3. *Let G be a group acting holomorphically and effectively on a Riemann surface X, and fix a point $p \in X$.*

1. *If the stabilizer subgroup G_p is finite, then G_p is a finite cyclic group.*
2. *If G is a finite group, then all stabilizer subgroups are finite cyclic subgroups.*
3. *If G is a finite group, the points of X with non-trivial stabilizers are discrete.*

We now need to find the complex charts of X/G and this is done with the following proposition.

Proposition 1.3.4. *Let G be a group acting holomorphically and effectively on a Riemann surface X, and fix a point $p \in X$. Then there is an open neighbourhood U of p such that*

1. *U is invariant under the stabilizer G_p. That is, $g \cdot u \in U$ for every $g \in G_p$ and $u \in U$.*
2. *$U \cap g(U) = \emptyset$ for every $g \notin G_p$.*
3. *The natural map $\alpha : U/G_p \to X/G$, induced by sending a point in U to its orbit, is a homeomorphism onto an open subset of X/G.*
4. *No point of U except p is fixed by any element of G_p.*

Now in order to define charts on X/G we first define charts on U/G_p and then transport these to X/G via the map α.
Choose a point $\bar{p} \in X/G$, so that $\bar{p}$ is the orbit of a point $p \in X$. Suppose that $|G_p| = 1$, so that the stabilizer of p is trivial. Then by the above proposition, there is a neighbourhood of U of p such that

$$\pi|_U : U \to W \subset X/G$$

is a homeomorphism onto a neighbourhood W of $\bar{p}$. By shrinking U if necessary, assume that U is the domain of the chart

$$\phi : U \to V$$

on X.

Then we can take as a chart on X/G the composition

$$\varphi = \phi \circ \pi|_U^{-1} : W \to V.$$

Since both ϕ and $\pi|_U$ are homeomorphisms, this is a chart on X/G.
To form a chart near a point $\bar{p}$ with $m = |G_p| \geq 2$ we must find an appropriate function from a neighbourhood of $\bar{p}$ to $\mathbb{C}$.
Again, using the proposition, choose a G_p-invariant neighbourhood U of p such that the map

$$\alpha : U/G_p \to W \subset X/G$$

is a homeomorphism onto a neighbourhood W of $\bar{p}$. Moreover, assume that the map $U \to U/G_p$ is exactly m-to-1 away from the point p.
We seek a mapping $\phi : W \to \mathbb{C}$ to serve as a chart near $\bar{p}$. The composition of such a map with α and the quotient map from U to U/G_p would be a G_p-invariant function

$$h : U \to U/G_p \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{C}$$

on a neighbourhood of p. ϕ is found by first finding h.
Let z be a local coordinate centered at p. For each $g \in G_p$, we have the function $g(z)$, which has multiplicity one at p. Define

$$h(z) = \prod_{g \in G_p} g(z).$$

Then h has multiplicity $m = |G_p|$ at p, and is defined on some G_p-invariant neighbourhood of p. Shrink U if necessary and assume h is defined on U. Clearly h is holomorphic and G_p invariant; applying an element of G_p simply permutes the factors in the definition of h.
Therefore, h descends to a continuous function

$$\bar{h} : U/G_p \to \mathbb{C}.$$

Finally, $\bar{h}$ is 1-to-1. To see this, note that the holomorphic map h has multiplicity m and hence is m-to-1 near p, so is the map from V to U/G_p away from p.
Since $\bar{h}$ is 1-to-1, continuous and open it is a homeomorphism and composing it with the inverse of $\alpha : U/G_p \to W$ gives a chart on W:

$$\phi : W \xrightarrow{\alpha^{-1}} U/G_p \xrightarrow{\bar{h}} V \subset \mathbb{C}$$

Theorem 1.3.5. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. The above construction of complex charts on X/G makes X/G into a Riemann surface. Moreover, the quotient map*

$$\pi : X \to X/G$$

is holomorphic of degree $|G|$, and $mult_p(\pi) = |G_p|$ for any point $p \in X$.

Ramification of the quotient map

Let G be a finite group acting holomorphically and effectively on a compact Riemann surface X, with quotient $X/G = Y$.
Suppose that $y \in Y$ is a branch point of the quotient map $\pi : X \to Y$. Let $x_1, \ldots, x_s$ be the points of X lying above y. These points form a single orbit for the action of G on X.
Since the x_i's are all in the same orbit, they have conjugate stabilizer subgroups, and in particular, each stabilizer subgroup is of the same order, say r.
Moreover, the number s of points in the orbit is the index of the stabilizer, and so is equal to $|G|/r$. This proves the following.

Lemma 1.3.6. *Let G be a finite group acting holomorphically and effectively on a compact Riemann surface X, with quotient map*

$$\pi : X \to Y = X/G.$$

Then for every branch point $y \in Y$ there is an integer $r \geq 2$ such that $\pi^{-1}(y)$ consists of exactly $|G|/r$ points of X, and at each of the pre-image points, π has multiplicity r.

Using this we get another way of writing the Riemann-Hurwitz formula.

Corollary 1.3.7. *Let G be a finite group acting holomorphically and effectively on a compact Riemann surface X, with quotient map $\pi : X \to Y = X/G$. Suppose that there are k branch points $y_1, \ldots, y_k$ in Y, with π having multiplicity r_i at the $|G|/r_i$ points above y_i. Then*

$$\begin{aligned} 2g_X - 2 &= |G|\Big(2g_{X/G} - 2\Big) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1) = \\ &= |G|\Big[2g_{X/G} - 2 + \sum_{i=1}^{k} \big(1 - \frac{1}{r_i}\big)\Big] \end{aligned}$$

Monodromy

Covering spaces and the fundamental group

It is known that compact orientable 2-manifolds[4] are classified (topologically) by their genus. This is best pictured using the fundamental group.
The standard polygon for the genus g surface has a boundary path of the form

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1},$$

where successive letters denote successive edges and edges with negative exponents have opposite direction. Edges with the same letter are pasted

[4] I.e. surfaces.

together, with their arrows matching. The general picture is shown in the below figure.

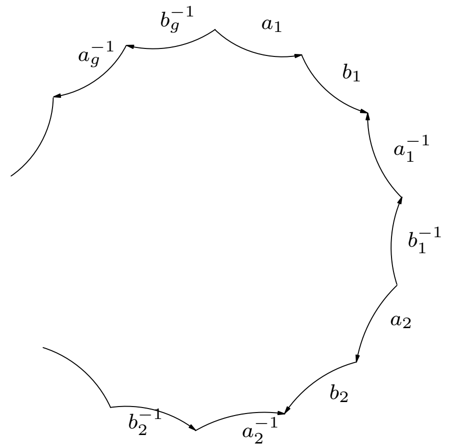


Figure 1.6: Picture of the side-pairing of the standard polygon for a genus g surface

This gives that the fundamental group $\pi_1(X, x_0) = \pi_1(X)$ is generated by $2g$ standard loops $\{A_1, B_1, \ldots, A_g, B_g\}$ such that

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = 1$$

Thus, the fundamental group is the set of homotopy classes of loops based at some point $x_0 \in X$ and the group structure is given by concatenation of such loops. A connected space is called **simply connected** if its fundamental group is trivial.

Definition 1.3.8. A **covering** of a Riemann surface X is a continuous map $F : Y \to X$ such that F is onto, and for every point $p \in X$ there is a neighbourhood W of p such that $F^{-1}(W)$ is a disjoint union of open subsets U_α, each mapping homeomorphically (via F) onto W. The pair (Y, F) is called a **covering space** of X and the number of components of $F^{-1}(W)$ is called **the number of sheets** of the covering.

Remark 1.3.9. *The above definition of coverings is sometimes referred to as* ***smooth coverings*** *as opposed to* ***branched coverings*** *which arise when $F : Y \to X$ is a nonconstant holomorphic map. These maps will be discussed later when we deal with the holomorphic maps.*
Also the covering space Y is assumed to be connected.

Definition 1.3.10. If $F : Y \to X$ is a covering, then a homeomorphism $G : Y \to Y$ is called a **deck transformation** (or **cover transformation** or

automorphism) if $G \circ F = F$. Under composition, these homeomorphisms form a group called the **group of deck transformations** (or **group of automorphisms**) denoted by $G(Y/X)$ or $Aut(Y/X)$.

Some properties of coverings:

1. Two coverings $F_1 : Y_1 \to X$ and $F_2 : Y_2 \to X$ are isomorphic if there is a homeomorphism $G : Y_1 \to Y_2$ such that $F_2 \circ G = F_1$.

2. There exists a universal cover $F_0 : Y_0 \to X$ such that Y_0 is simply connected and F_0 is unique up to isomorphism. If $F : Y \to X$ is any other covering of X then there is a unique covering $G : Y_0 \to Y$, factoring F_0 uniquely, such that $F_0 = F \circ G$.

3. Coverings $F : Y \to X$ preserves paths by the path-lifting property, that is, for any path $\gamma : [0, 1] \to X$ and any pre-image p of $\gamma(0)$ there is a path $\tilde{\gamma}$ on Y such that $\tilde{\gamma}(0) = p$ and $F \circ \tilde{\gamma} = \gamma$.

4. *(The monodromy theorem).* Let $F : Y \to X$ be a covering and let γ_1 and γ_2 be paths in X starting and ending at the same point such that γ_1 is homotopic to γ_2. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be the lifts of γ_1 and γ_2, respectively, starting at the same point above $\gamma_1(0)$. Then $\tilde{\gamma}_1$ is homotopic to $\tilde{\gamma}_2$, and in particular $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$.

The fundamental group $\pi_1(X, x_0)$ acts on the universal cover $F_0 : Y_0 \to X$ by sending points $u \in Y_0$ to a point $[\gamma] \cdot u \in X$, where $[\gamma] \cdot u$ is a point depending only on u and the homotopy class $[\gamma]$ of some loop γ.
This action of $\pi_1(X, x_0)$ on the universal cover, preserves the fibres of the covering map F_0, and the orbit space $Y_0/\pi_1(X, x_0)$ is homeomorphic to the space X.
Now, each covering $F : Y \to X$ corresponds to a subgroup D of $\pi_1(X, x_0)$, where D is the subset of paths $[\gamma] \in \pi_1(X, x_0)$ such that the lift of γ starting at $y_0 \in Y$ is a closed path. The fact that D is a well defined subgroup of the fundamental group comes from the monodromy theorem.
In fact, for a covering $F : Y \to X$ and with $x_0 \in X$, $y_0 \in Y$ we can define $f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ by $f_*([\gamma]) = [F \circ \gamma]$ and then the corresponding subgroup becomes $D = f_*(\pi_1(Y, y_0)) \subseteq \pi_1(X, x_0)$.
Thus, we can put a partial ordering of the covering spaces corresponding to their covering groups $D \subseteq \pi_1(X, x_0)$ and with the above action of D restricted to the universal covering space Y_0 gives an orbit space Y_0/D, which is a covering space of X. In fact, every connected covering space of X occurs in this way and two orbit spaces Y_0/D and Y_0/D' are isomorphic if and only if D and D' are conjugate subgroups of the fundamental group. This gives the 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{connected coverings} \\ F : Y \to X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes} \\ \text{of subgroups} \\ D \subseteq \pi_1(X, x_0) \end{array} \right\}$$

The **degree** of a covering is the number of pre-images of a point of X and this is exactly the index of the subgroup D inside the fundamental group.

Example 1.3.11. Let $X = \mathbb{C}/\Lambda$ for some lattice Λ. This is a Riemann surface (a complex torus) of genus 1, and the natural quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda = X$ is the universal cover of X. The fundamental group is a free Abelian group of two generators, isomorphic to the lattice Λ. Moreover, the action of Λ on the universal cover $\mathbb{C}$ is by translations.

If $F : Y \to X$ is a covering, and $D = f_*(\pi_1(Y, y_0))$ then, the group of deck transformations, $G(Y/X)$, is

$$G(Y/X) \cong N(D)/D$$

where $N(D)$ is the normalizer of D in $\pi_1(X, x_0)$. If D is a normal subgroup of the fundamental group of X, then $G(Y/X) \cong \pi_1(X, x_0)/D$. Such coverings are called **regular coverings**. Every universal covering $F : Y_0 \to X$, is regular with group of deck transformations isomorphic to $\pi_1(X, x_0)$.

Monodromy of a finite covering

Let $F : Y \to X$ be a finite (smooth) covering of degree n. This means that all points of X have exactly n pre-images in Y and the covering F corresponds to a subgroup $D \subseteq \pi_1(X, x_0)$ such that $|\pi_1(X, x_0) : D| = n$. Now the fiber $F^{-1}(x_0)$ over x_0 consists of n points $\{y_1, \dots, y_n\}$ lying over x_0. If we consider a loop γ in X based at x_0 then the lifting of this loop to Y will become n paths $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$, each $\tilde{\gamma}_i$ being the unique lift of γ starting at y_i, that is $\tilde{\gamma}_i(0) = y_i$

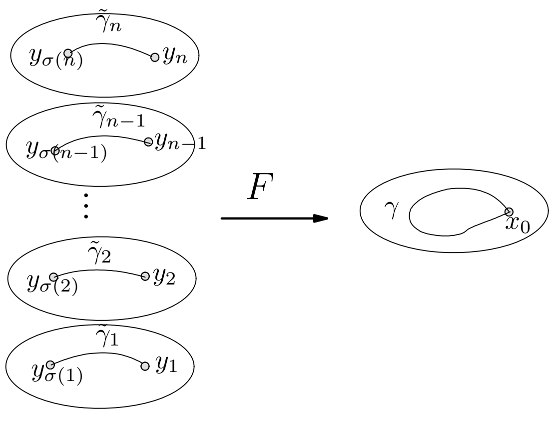


Figure 1.7: The lifting of loops in a finite smooth covering. If the covering is not regular there may appear closed pre-images of loops.

What about the end points $\tilde{\gamma}_i(1)$? These also lie above x_0 and also form the entire pre-image set $F^{-1}(x_0)$. Thus, each $\tilde{\gamma}_i(1)$ is some y_j for some j and so we denote this by $y_{\sigma(i)}$.
Thus, σ is a permutation of the indices $\{1, \ldots, n\}$ and by the monodromy theorem, this is well defined and depends only on the homotopy class of γ. Therefore it is natural to get a group homomorphism

$$\varphi : \pi_1(X, x_0) \to \Sigma_n$$

where Σ_n is the symmetric group of all permutations on $\{1, \ldots, n\}$.

Definition 1.3.12. The monodromy representation of a finite covering $F : Y \to X$ is the group homomorphism $\varphi : \pi_1(X, x_0) \to \Sigma_n$ defined above.

An illustrative way of seeing how the monodromy representation works is to prove the following lemma.

Lemma 1.3.13. *Let $\varphi : \pi_1(X, x_0) \to \Sigma_n$ be the monodromy representation of a finite covering map $F : Y \to X$ of degree n, with Y connected. Then the image of φ is a transitive subgroup of Σ_n.*

Proof. With the same notation as above, consider two points y_i and y_j in the fiber of F over x_0. Since Y is connected, there is a path $\tilde{\gamma}$ in Y starting in y_i and ending in y_j. Let $\gamma = F \circ \tilde{\gamma}$ be the image of $\tilde{\gamma}$ in X. γ is then a loop in X based at x_0, since both y_i and y_j map to x_0 under F. So we get from the construction of φ that the image $\varphi([\gamma])$ is a permutation which sends i to j. □

Definition 1.3.14. The transitive subgroup of Σ_n defined above is called the **monodromy group** of the covering $F : Y \to X$ and is denoted by $M(Y/X)$.

Monodromy of a holomorphic map

If we now turn our attention towards holomorphic maps between Riemann surfaces[5] and try to apply what we have seen about covering spaces, fundamental groups and monodromy representation, what can we say about these?
First of all, a holomorphic nonconstant map $F : Y \to X$ need not be a covering map at all. This is because branching may occur, and this means that from the definition of a covering map, things will be a bit more complicated.
The solution is to view the holomorphic map as a smooth covering by removing all the branch points and their respective ramification points. Let $R \subset Y$ be the finite set of ramification points of F, and let $B = F(R) \subset X$ be the set of branch points. Define $X^* = X \backslash B$ and $Y^* = Y \backslash F^{-1}(B)$.

[5] That is, branched coverings.

Thus, X^* and Y^* are the original spaces with the branch points removed, respectively all points mapped to branch points by F removed. The latter need not all be ramification points.

Now, for each $x \in X^*$ the pre-image $F^{-1}(x)$ consists of n distinct points, each having multiplicity one. Hence the restriction $F|_{Y^*} : Y^* \to X^*$ is indeed a covering map of degree n.

This covering corresponds to the monodromy representation

$$\varphi : \pi_1(X^*, x) \to \Sigma_n$$

and this is the monodromy representation of the holomorphic map F. Since Y is connected, so is Y^* and thus the image of φ is a transitive subgroup of Σ_n.

For each branch point $b \in X$, choose a small open neighbourhood W of b in X. The punctured open set $W \backslash \{b\}$ is an open subset of X^* and is isomorphic to a small punctured disc. Denote the pre-images of b by $u_1, u_2, \ldots, u_k \in Y$. Note that the number k of such points is less than the degree n of F since b is a branch point so at least one of the u_j's must have multiplicity greater than 1.

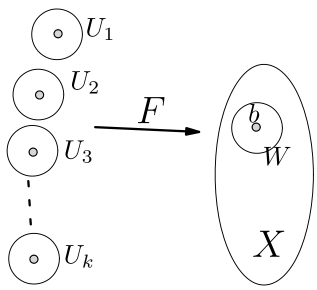


Figure 1.8: The pre-images of a branched point

Choosing W small enough so that F^{-1} consists of disjoint open neighbourhoods $U_1, U_2, \ldots, U_k$ of the points $u_1, u_2, \ldots, u_k$. Set $m_j = \mathrm{mult}_{u_j}(F)$ to be the multiplicity of F at each u_j. Then, locally each U_j has coordinates z_j on U_j and z on W such that locally the map F has the form $z = z_j^{m_j}$ on U_j.

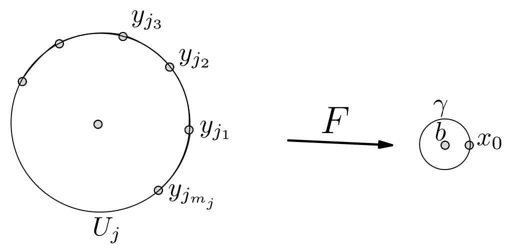


Figure 1.9: The ramification of the pre-images

We can see how small loops around the branch point is lifted up to m_j paths which together form a loop around each u_j. So the cycle structure representing a small loop around b becomes $(m_1)(m_2)\dots(m_k)$, that is each cycle is of length m_j.

The Euler characteristic of a Riemann surface

The complex structure of a Riemann surface provides a geometrical structure as a 2-orbifold. We can extend the definition of the Euler characteristic to Riemann surfaces with cone points.

Definition 1.3.15. Let X be a Riemann surface of genus g with branch points $b_1, \dots, b_r$ having orders $m_1, \dots, m_r$. Then the Euler characteristic is given by

$$\chi(X) = 2 - 2g - \sum_{i=1}^{r} \big(1 - \frac{1}{m_i}\big)$$

Using this definition and together with branched coverings, we can now state the well-known version of the Riemann-Hurwitz theorem.

Theorem 1.3.16. *Let $F : Y \to X$ be an n-sheeted branched covering (holomorphic map) between compact Riemann surfaces X and Y. Then*

$$\chi(Y) = n \cdot \chi(X) \tag{1.1}$$

That is, under coverings the Euler characteristic is multiplicative.

Uniformization

We have seen that for any connected manifold X there is always a universal covering $F : Y \to X$, where Y is a simply connected manifold. Also, if $F^{'} : Y' \to X$ is any other covering, with $p \in X$, $q \in Y$ and $q^{'} \in Y^{'}$ such that $F(q) = p = F^{'}(q^{'})$, then there is a unique covering map $f : Y \to Y^{'}$ such that $f(q) = q^{'}$ and $F = F^{'} \circ f$.
Applying the theory of coverings to Riemann surfaces yields the following proposition.

Proposition 1.3.17. *If X is a Riemann surface and $F : Y \to X$ is a universal covering of X, then Y is a simply connected Riemann surface and F is holomorphic and unramified.*

Since the types of simply connected Riemann surfaces are known one can show that:

Theorem 1.3.18. *(Main theorem of uniformization) Every simply connected Riemann surface is isomorphic to either $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$.*

This now gives us the useful theorem:

Theorem 1.3.19. *Every Riemann surface X is homeomorphic to some quotient space Y/G, where Y is the universal covering space and G is the covering group ($\subset Aut(Y)$) consisting of all $\gamma \in Aut(Y)$ with $F \circ \gamma = F$ permuting the fibres of F.*

A natural question is what the groups of conformal automorphisms are for the three simply connected Riemann surfaces $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$? We have seen that the automorphisms of a Riemann Surface X are defined as the isomorphisms $F : X \to X$ and that these automorphisms form a group under composition denoted $Aut(X)$.

(1). Automorphisms of the complex plane $\mathbb{C}$ are simply transformations of the form $z \mapsto az + b$ for $a \neq 0$, that is

$$\mathrm{Aut}\,(\mathbb{C}) = \{z \mapsto az + b | a \in \mathbb{C}^*, b \in \mathbb{C}\}$$

(2). The automorphisms of the Riemann sphere, $\widehat{\mathbb{C}}$, are Möbius transformations with complex coefficients, namely

$$z \mapsto \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})/\pm I$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Such transformations can be described using 2×2-matrices and with the use of conventional notation these matrices form a group called the **projective special linear group** denoted $PSL_2(\mathbb{C})$.

(3). The automorphisms of the hyperbolic plane $\mathbb{H}$ ($\cong \mathbb{D}$) are also Möbius transformations, more precisely they are a subgroup of the Möbius group that sends the extended real axis $\widehat{\mathbb{R}}$ to itself and preserves the upper and lower half-planes.

$$z \mapsto \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})/\pm I$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Thus the automorphism group is $\mathrm{Aut}\,(\mathbb{H}) = PSL_2(\mathbb{R})$.

So what can we say about the group G acting on the simply connected space Y? First of all, G acts without fixed points, is torsion free and acts discontinuously. Moreover, since there are only three simply connected spaces we get the following three cases.

(1). If $Y = \mathbb{C}$ then $\mathrm{Aut}\,(\mathbb{C}) = \{z \mapsto az + b | a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C}\}$. A map $z \mapsto az + b$ has no fixed point if and only if $a = 1$. This means that G is a group of translations and hence G is either isomorphic to $\mathbb{Z}$, a lattice Λ or $\{id\}$.

Thus, $X = \mathbb{C}/\mathbb{Z}$, $X = \mathbb{C}/\Lambda$ or $X = \mathbb{C}$. In the case of $X = \mathbb{C}/\Lambda$, then as we have seen X is a torus which also may be regarded as an elliptic curve of genus 1.

(2). If $Y = \widehat{\mathbb{C}}$ then $\mathrm{Aut}\,(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C})$. Now any element of $\mathrm{Aut}\,(\widehat{\mathbb{C}})$, say $z \mapsto \frac{az+b}{cz+d}$ acting on $\widehat{\mathbb{C}}$ must have fixed points, and so the only possibility is for $G = \{id\}$.

Hence $X = \widehat{\mathbb{C}}$.

(3). In all other cases of G, $Y = \mathbb{H}$, in particular for all compact Riemann surfaces[6] with genus $g > 1$. In this case, $G \subset \mathrm{Aut}\,(\mathbb{H}) = PSL_2(\mathbb{R})$ and is what is called a Fuchsian group.

The case with $g > 1$ is the general and most interesting case, so in order to study this we will first need to define Fuchsian groups.

Fuchsian groups

The theory of Fuchsian groups is well described in the books [5],[25] and [34]. The following is simply a highlight of the topics of Fuchsian groups.

We have seen that the group $PSL_2(\mathbb{R})$ is the group of automorphisms of the hyperbolic plane $\mathbb{H}$ and the elements of this group are transformations of the form

$$z \mapsto \frac{az+b}{cz+d}$$

$a, b, c, d \in \mathbb{R}$ and $ad-bc = 1$. This group can also be made into a topological space by identifying the transformation with the point $(a, b, c, d) \in \mathbb{R}^4$ in the subspace $\{(a, b, c, d) \in \mathbb{R}^4 | ad - bc = 1\}$ and in fact one can show that $PSL_2(\mathbb{R})$ is homeomorphic to $\mathbb{R}^2 \times S^1$ and thus being a 3-dimensional manifold, moreover the group multiplication and taking of inverses are continuous with the topology on $PSL_2(\mathbb{R})$ and so this is a topological group.

Definition 1.3.20. A **Fuchsian** group Γ is a *discrete*[7] subgroup of $PSL_2(\mathbb{R})$.

[6] Without singular points.

[7] A subgroup G of a topological group is discrete if and only if the subspace topology on G is the discrete topology

As we deal with $PSL_2(\mathbb{R})$ the notion of discreteness can be defined using the norm of the corresponding matrix in the following way.

Definition 1.3.21. A subgroup Γ of $PSL_2(\mathbb{R})$ is discrete if and only if for each positive k, the set $\{A \in \Gamma \mid \|A\| \leq k\}$ is finite.[8]

Example 1.3.22. An example of a Fuchsian group is the **modular group** $PSL_2(\mathbb{Z})$, the subgroup of $PSL_2(\mathbb{R})$ consisting of matrices with integer elements. It is easy to see with the above definition that this group is discrete.

Fuchsian groups and their action on the hyperbolic plane $\mathbb{H}$ are related to lattice groups and their action on the Euclidean plane. The lattice groups are discrete groups of Euclidean isometries and their quotients are compact Riemann surfaces homeomorphic to the torus. On the other hand, Fuchsian groups are discrete groups of hyperbolic geometries and their quotient spaces are Riemann surfaces of genera $g > 1$.
There are two methods for the construction of Fuchsian groups

- The arithmetic way is to construct Fuchsian groups as discrete subgroups of $PSL_2(\mathbb{R})$ by means of number theory. Such an example is $PSL_2(\mathbb{Z})$.
- The geometric way (which in fact is the same as Poincaré's uniformization) is to start with a fundamental domain[9] F for G and then generate G by side-pairing transformations of F. Examples of this construction are the triangle groups.

Lattices have an interesting property in that their action on $\mathbb{C}$ is free. Fuchsian groups do not have this property in general, they instead have a slightly weaker property called *properly discontinuous action.*[10]

Definition 1.3.23. A group G acts **properly discontinuously** in Y if for all $y \in Y$ there is a neighbourhood U such that for all $g \in G$,

$$g(U) \cap U = \emptyset$$

except for finitely many $g \in G$.

Theorem 1.3.24. *Let Γ be a subgroup of $PSL_2(\mathbb{R})$. Then Γ is a Fuchsian group if and only if Γ acts discontinuously on $\mathbb{H}$.*

Note that if a group G acts properly discontinuously on a space Y, then the stabilizer of any point of Y is finite. As we have seen, when the action is free then all the stabilizers are trivial.
The elements of a Fuchsian group can either be *parabolic, elliptic* or *hyperbolic* and this coincides with the definition of parabolic, elliptic and hyperbolic isometries in chapter 1.1.

[8] The norm of a matrix $A \in PSL_2(\mathbb{R})$ is defined as $\|A\| = (a^2 + b^2 + c^2 + d^2)^{1/2}$.
[9] A in a sense suitable hyperbolic polygon
[10] Some books refer to free actions as discontinuous actions and then calls the weaker action for properly discontinuous.

The structure of Fuchsian groups

A Fuchsian group Γ has a presentation with generators

$$x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g, p_1, \ldots, p_s, h_1, \ldots, h_t$$

and relations

$$x_1^{m_1} = \ldots = x_r^{m_r} = \prod_{i=1}^{r} x_i \prod_{j=1}^{g} [a_j, b_j] \prod_{k=1}^{s} p_k \prod_{l=1}^{t} h_l = 1,$$

where $[a_j, b_j] = a_j b_j a_j^{-1} b_j^{-1}$. The last relation is called the long relation. There are 4 types of generators for a Fuchsian group.

1. x_i are the elliptic elements.

2. a_j, b_j are the hyperbolic elements.

3. p_k are the parabolic elements.

4. h_l are the hyperbolic boundary elements.

This information is encoded in the signature of a Fuchsian group, and this is defined for a Fuchsian group Γ with the above presentation to be

$$s(\Gamma) = (g; m_1, \ldots, m_r; s; t) \tag{1.2}$$

where $m_i \geq 2$ are integers and are called the periods of Γ. The integer g is the genus of the underlying surface of the Fuchsian group $(\mathbb{H}/\Gamma)$ and since we are only interested in compact Riemann surfaces, we will turn our attention towards Fuchsian groups without parabolic elements and hyperbolic boundary elements. Such signatures will be written $(g; m_1, \ldots, m_r)$.

The action of an elliptic element x_i in a Fuchsian group which satisfies the relation $x_i^{m_i}$, corresponds to a rotation of angle $2\pi/m_i$ around a point in the hyperbolic plane $\mathbb{H}$. In fact, any elliptic element in Γ is conjugate to some power of x_i for $i = 1, \ldots, r$.[11]

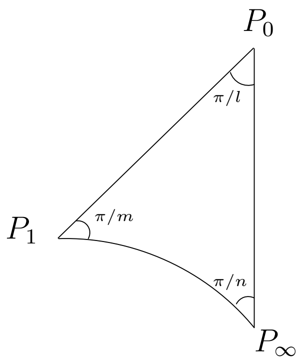
A Fuchsian group Γ containing only hyperbolic elements is called a **Fuchsian surface group** and its signature is $s(\Gamma) = (g; -)$.

[11] In fact, this holds for co-compact Riemann surfaces.

Example 1.3.25. Triangle groups

The triangle groups are groups Δ with signature $s(\Delta) = (0; l, m, n)$ [12]. Define a triangle in $\mathbb{D}$ with angles $\pi/l, \pi/m$ and π/n for $l, m, n \in \mathbb{N}\backslash\{0\}$ or ∞ satisfying

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1.$$



Let $\gamma_0, \gamma_1, \gamma_\infty$ be hyperbolic counterclockwise rotations around P_0, P_1, P_∞ with angles $2\pi/l, 2\pi/m, 2\pi/n$ the above angles.
Then they generate a triangle group $\Delta = (0; l, m, n)$ with presentation

$$\langle \gamma_0, \gamma_1, \gamma_\infty | \gamma_0^l = \gamma_1^m = \gamma_\infty^n = \gamma_0\gamma_1\gamma_\infty = 1 \rangle$$

acting discontinuously on the hyperbolic plane $\mathbb{H}$. The fundamental domain of Δ can be given as the union of the triangle $P_0P_1P_\infty$ with a triangle constructed by hyperbolic reflection in one side.

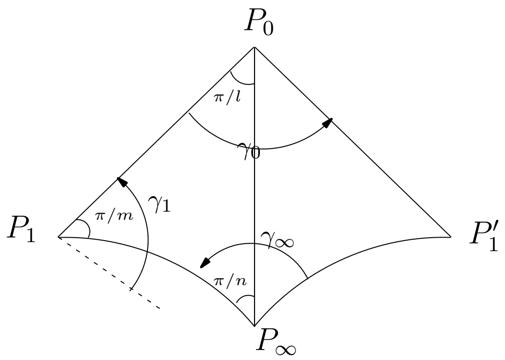


Figure 1.10: The fundamental domain for a triangle group $(0; l, m, n)$.

If any of the vertices of the triangle, say P_l, lies on the boundary of $\mathbb{H}$ the angle is of course 0 and in that case we define $l = \infty$ and omit the relation $\gamma_0^l = 1$ from the presentation. Then γ_0 becomes a parabolic element with fixed point P_l. This means the quotient space $\mathbb{H}/\Delta$ is no longer compact.

[12] Usually these groups are written as $\langle l, m, n \rangle$ but with our notation they are groups with signature $(0; l, m, n)$.

The fundamental domain for a Fuchsian group

As with triangle groups we can arrange some fundamental domain in $\mathbb{H}$ corresponding to a Fuchsian group Γ. In this case it is a bit more complicated than in the case of triangle groups.

Definition 1.3.26. A domain D of the hyperbolic plane is a **fundamental domain** for a Fuchsian group Γ if and only if

1. given $z \in \mathbb{H}$ there exists $g \in \Gamma$ such that $g(z) \in D$.
2. if there exist $z \in D$ and $1_d \neq g \in \Gamma$ such that $g(z)$ is also in D, then $z, g(z) \in \partial D$.
3. the hyperbolic area of ∂D is 0, that is $\mu(\partial D) = 0$.

It is obvious that with such a definition, a fundamental domain is not unique. However, for a finitely generated Fuchsian group we can choose a fundamental domain such that the following holds:

1. The fundamental domain D is homeomorphic to a disc and such that ∂D is a union of h-segments, called the sides.
2. There is a finite number of points in ∂D which divide ∂D into these segments. These are the corners of the fundamental domain.

In this case, ∂D is called a **fundamental polygon** of Γ.
Two sides $\alpha, \alpha' \in \partial D$ are congruent if there exists a $g \in \Gamma$ such that $\alpha = D \cap g(D)$, $\alpha' = D \cap g^{-1}(D)$ and $g(\alpha) = \alpha'$. If g is an involution, then α and α' lie on the same h-line.
If $D \cap g(D) \neq \emptyset$ for some $g \in \Gamma$ and $D \cap g(D)$ is not a common side, then $D \cap g(D)$ consists of a finite number of vertices.
Thus, conjugated vertices in D correspond to a either a cone point in $\mathbb{H}/\Gamma$, when the sum of the angles is $2\pi/m_i$ or to a regular point with angle sum 2π.
Now the generators of the Fuchsian group Γ pair the congruent sides of a fundamental polygon for Γ. The configuration of one such polygon, named a **labelled polygon**, is for a Fuchsian group with signature $s(\Gamma) = (g; m_1, \ldots, m_r)$:

$$\gamma_1\gamma_1' \ldots \gamma_r\gamma_r'\alpha_1\beta_1\alpha_1'\beta_1' \ldots \alpha_g\beta_g\alpha_g'\beta_g'.$$

The generator x_i pairs the sides γ_i and γ_i', a_i pairs the sided α_i and α_i' and b_i pairs the sides β_i and β_i'.

Remark 1.3.27. *It is important to notice that the above construction implies that $\mathbb{H}/\Gamma$ is isomorphic to the fundamental domain together with the side pairing. In that way we can construct the Riemann surface X by the equivalence relation $\sim$ acting on the fundamental domain D and we get $X \cong D/\sim$.*

Example 1.3.28. Take the Fuchsian group Δ with signature $s(\Delta) = (0; 2, 2, 2, 3)$. This has a canonical presentation given by

$$\Delta = \langle x_1, x_2, x_3, x_4 | x_1^2 = x_2^2 = x_3^2 = x_4^3 = x_1x_2x_3x_4 = 1\rangle$$

Start of with a polygon with angles $\pi/2, \pi/2, \pi/2$ and $\pi/3$.

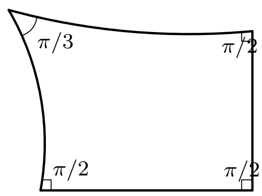


Figure 1.11: Hyperbolic polygon with angles $\pi/2, \pi/2, \pi/2$ and $\pi/3$.

In order to make this a fundamental polygon for the Fuchsian group we need to double it and then label the sides as in the below picture.

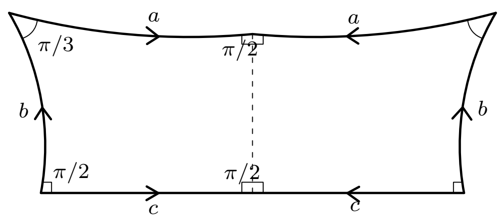


Figure 1.12: The pre-images of a branched point

Now the generator x_4 has order 3 and with a rotation around the corner P_4 of angle $2\pi/3$ it sends the edge b to the opposite edge b. Similarly for the other generators x_1, x_2 and x_3. Thus, we obtain the desired side-pairing from the generators of the Fuchsian group.

Thus for a Fuchsian group Γ and its corresponding fundamental domain, a Γ-tessellation of $\mathbb{H}$ is the configuration of $\mathbb{H}$ formed by D and its images under Γ.
The **hyperbolic area of a Fuchsian group** with signature (1.2) coincides with the area of any of its fundamental domains. By the Gauss-Bonnet formula, this area becomes

$$\mu(\Gamma) = 2\pi\Big(2g - 2 + \sum_{i=1}^{r} \big(1 - \frac{1}{m_i}\big)\Big) = -2\pi\chi(\mathbb{H}/\Gamma) \tag{1.3}$$

A fundamental domain D, with the above identifications on its boundary, has the structure of a compact Riemann surface.

The converse of this is also true and is known as Poincaré's theorem or Poincaré's theorem of uniformization.

Theorem 1.3.29. *(Poincaré's theorem) Given a fundamental polygon ∂D, satisfying the above conditions such that $\mu(D) > 0$, then there exists a Fuchsian group Γ with signature (1.2) having D as its fundamental domain.*

It is important to notice that the side-pairings of the fundamental polygon ∂D generate the Fuchsian group Γ.

Example 1.3.30. If any of the vertices of the triangle $P_l P_m P_n$, say P_l, lies on the boundary of $\mathbb{H}$ the angle is of course 0 and in that case we define $l = \infty$ and omit the relation $\gamma_0^l = 1$ from the presentation. Then γ_0 becomes a parabolic element with fixed point P_l. This means the quotient space $\mathbb{H}/\Delta$ is no longer compact. Furthermore, the area of the fundamental domain is infinite.
An example of such triangle group is the modular group $\langle 2, 3, \infty \rangle$ with signature $(0; 2, 3; 1)$. Its fundamental domain is given in $\mathbb{H}$ in the below picture.

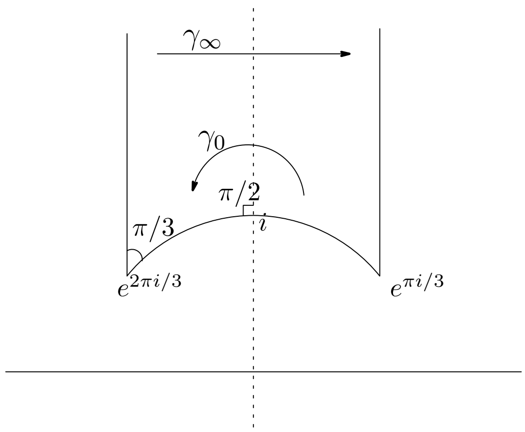


Figure 1.13: The fundamental domain of the triangle group $\langle 2, 3, \infty \rangle$.

and this is in fact the discrete subgroup $PSL_2(\mathbb{Z})$ of $PSL_2(\mathbb{R})$.

Fuchsian subgroups

Let Γ' be a subgroup of Γ of index N. Then $\Gamma = \cup_1^N g_i(\Gamma')$, where $\{g_i\}$ are the transversals of Γ' in Γ. Then, if D is a fundamental domain for Γ, we get that

$$D' = \bigcup_{i=1}^{N} g_i(D)$$

is a fundamental domain for Γ'.
That is, the monomorphism $i : \Gamma' \to \Gamma$ determines, via the transitive permutation representation $\theta : \Gamma \to \Sigma_N$, the covering $f : \mathbb{H}/\Gamma' \to \mathbb{H}/\Gamma$.
The map $\theta : \Gamma \to \Sigma_N$ is called the **monodromy** of the covering f and $\theta(\Gamma)$ is the **monodromy group** of the covering.
Given the above, one can calculate the structure of Γ' in terms of the structure of Γ and the action of Γ on the Γ'-cosets. This was done by Singerman for general Riemann surfaces, but here we state a version for compact Riemann surfaces. (See [37]).

Theorem 1.3.31. *([37]) Let Γ be a Fuchsian group with signature $s(\Gamma) = (0; m_1, \ldots, m_r)$ and a canonical representation. Then Γ contains a Fuchsian subgroup Γ' of index N with signature*

$$s(\Gamma') = (h; m'_{1,1}, m'_{1,2}, \ldots, m'_{1,s_1}, \ldots, m'_{r,1}, \ldots, m'_{r,s_r})$$

if and only if there exists a transitive permutation representation $\theta : \Gamma \to \Sigma_N$ satisfying the following conditions:

1. *The permutation $\theta(x_i)$ has precisely s_i cycles of lengths less than m_i, the lengths of these cycles being $m_i/m'_{i,1}, \ldots, m_i/m'_{i,s_i}$.*
2. *The Riemann-Hurwitz formula*

$$\frac{\mu(\Gamma')}{\mu(\Gamma)} = N$$

is satisfied.

The quotient space $\mathbb{H}/\Gamma$

Clearly from theorem 1.3.5 the quotient space of $\mathbb{H}$ with a Fuchsian group Γ is a Riemann surface and the natural projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is a holomorphic map.
The importance of Fuchsian surface groups is made apparent from the following theorem which is a start of our classification of Riemann surfaces. This, as we shall see, is generally done by using Teichmüller's theory, but for now here is a preview of the upcoming theory.

Theorem 1.3.32. *[25] Let Γ_1 and Γ_2 be two Fuchsian surface groups. Then $\mathbb{H}/\Gamma_1$ and $\mathbb{H}/\Gamma_2$ are conformally equivalent if and only if there exists an element $\tau \in PSL_2(\mathbb{R})$ such that $\tau\Gamma_1\tau^{-1} = \Gamma_2$.*

Theorem 1.3.33. *[25] If Γ is a Fuchsian surface group then*

$$Aut(\mathbb{H}/\Gamma) \cong N(\Gamma)/\Gamma$$

where $N(\Gamma)$ is the normalizer of Γ in $PSL_2(\mathbb{R})$.

In fact, if Γ is a non-cyclic Fuchsian group, then its normalizer in $PSL_2(\mathbb{R})$ is also a Fuchsian group.

Corollary 1.3.34. *[25] If Γ is a non-cyclic Fuchsian surface group, then every group of automorphisms of $\mathbb{H}/\Gamma$ is isomorphic to Δ/Γ where Δ is a Fuchsian group such that $\Gamma \trianglelefteq \Delta$.*

Generally, finite index subgroups of Fuchsian surface groups are again Fuchsian surface groups.
As we shall see, these results play an important role in finding the group of automorphisms of compact Riemann surfaces.

Theorem 1.3.35. *Every compact Riemann surface X_g, of genus $g \geq 2$, is conformally equivalent to a Riemann surface $\mathbb{H}/\Gamma$, uniformized by a Fuchsian surface group Γ.*

Automorphism groups of compact Riemann surfaces

We have seen that the groups of automorphisms of $\mathbb{C}, \widehat{\mathbb{C}}$ and $\mathbb{H}$ consists of Möbius transformations and that the group of automorphisms is not necessarily finite. However, for compact Riemann surfaces X_g of genus $g \geq 2$ this is always the case.

Proposition 1.3.36. *(Hurwitz' Theorem)[25] Let X be a compact Riemann surface of genus $g \geq 2$. Then*

$$|Aut(X)| \leq 84(g-1). \tag{1.4}$$

Proof. Let $X = \mathbb{H}/\Gamma$ where Γ is a Fuchsian surface group. Then $s(\Gamma) = (g; -)$ and the area for a fundamental domain for Γ is given by $\mu(\Gamma) = 2\pi(2g-2)$. Now, $\mathrm{Aut}\,(X) \cong \Delta/\Gamma$ for some Fuchsian group Δ such that $\Gamma \trianglelefteq \Delta$. From the compactness of $X = \mathbb{H}/\Gamma$ and the regular covering $\mathbb{H}/\Gamma \to \mathbb{H}/\Delta$ it follows that $\mathbb{H}/\Gamma$ is also compact and so has a fundamental region of finite area. It is easy to show, by looking at possible arrangements of hyperbolic triangles, that this area is bounded below by $\mu(\Delta) \geq \pi/21$. Hence it follows that

$$|\mathrm{Aut}\,(X)| = |\Delta : \Gamma| = \frac{\mu(\Gamma)}{\mu(\Delta)} = \frac{2\pi(2g-2)}{\mu(\Delta)} \leq 84(g-1).$$

□

Another way of stating whether a group is a group of automorphisms of a given Riemann surface is by using an epimorphism θ such that the epimorphism gives the desired relationship between the surface group Γ and the corresponding normalization of Γ.

Proposition 1.3.37. *Given a Riemann surface X represented as the orbit space $\mathbb{H}/\Gamma$, with Γ a Fuchsian surface group, a finite group G is a group of automorphisms of X if and only if there exists a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma$.*

$$\begin{array}{cccc} \theta: & \Delta & \xrightarrow{\theta} & G \\ & \downarrow & & \downarrow \\ & \Gamma & \longrightarrow & 1_d \end{array}$$

The Fuchsian group Δ is the lifting of G to the universal covering $f_u : \mathbb{H} \to \mathbb{H}/\Gamma$ and is called the **universal covering transformation group** of (X, G). Hence we have that the orbit space X/G is uniformized by Δ, $X/G = \mathbb{H}/\Delta$.

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & X & \longrightarrow & X/G \\ & & (\mathbb{H}/\Gamma) & & (\mathbb{H}/\Delta) \\ \updownarrow & & \updownarrow & & \updownarrow \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Delta \end{array}$$

The covering $f : X \to X/G = \mathbb{H}/\Delta$ is a regular branched covering with monodromy group $G = \Delta/\Gamma$.

Example 1.3.38. Hurwitz groups and Hurwitz surfaces

Automorphism groups of surfaces which obtain the maximal of the bound in equation (1.4) above are named **Hurwitz groups** after the mathematician Adolf Hurwitz (1859-1919) who proved the bound in 1893. The corresponding surfaces are called **Hurwitz surfaces** and these surfaces form an interesting category of surfaces in that they are the surfaces admitting a maximal number of automorphisms of a given genus.
There is a nice way of defining Hurwitz groups using the order of its generators.

Theorem 1.3.39. *[25] A finite group H is a Hurwitz group if and only if H is non-trivial and has two generators x and y satisfying the relations*

$$x^2 = y^3 = (xy)^7 = 1$$

From the proof of theorem (1.3.36) we saw that maximizing the automorphism group is the same as minimizing the area of some triangle group Δ. This minimal area $(\pi/21)$ is obtained by the unique triangle group $(0; 2, 3, 7)$ given by the following presentation.

$$\Delta = \langle X, Y, Z | X^2 = Y^3 = Z^7 = XYZ = 1 \rangle$$

It is a well known fact (and not hard to prove) that no Riemann surface of genus 2 admits an automorphism of order 7 and so there cannot be any Hurwitz surfaces of genus 2.
In the case of genus 3, there is a unique Hurwitz surface, namely the Klein quartic. Its automorphism group is $PSL_2(7)$. The equation of the Klein quartic is given by

$$x^3y + y^3z + z^3x = 0$$

in complex projective coordinates.
The next Hurwitz surface is the Macbeath curve, of genus 7, having $PSL_2(8)$ as automorphism group where the order is 504.
There are many more but we will not list them all in this thesis.

1.4 Teichmüller theory

This chapter is a short survey of the theory of Teichmüller theory. For further reading see [33] and [29].
In the following we will use the ordinary notations from complex analysis. That is $z = x + iy$, $dz = dx + idy$ and $d\bar{z} = dx - idy$. Also we write

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

When there is no possibility of confusion we will write $\partial w/\partial z = w_z$ as ∂w and $\partial w/\partial \bar{z} = w_{\bar{z}}$ as $\bar{\partial} w$.

Quasiconformal mappings

We begin with a smooth ($\mathcal{C}^\infty$) oriented 2-manifold X equipped with a $\mathcal{C}^\infty$ Riemann metric. In local smooth coordinates this metric can be expressed as

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2 = \lambda^2(z)|dz + \mu(z)d\bar{z}|^2$$

where $\lambda(z) > 0$ and $\mu(z)$ is complex valued with $|\mu(z)| < 1$.
Now, when is a $\mathcal{C}^1$ diffeomorphism $w(x, y) = u(x, y) + v(x, y)$ from an open subset on X to a domain in the w plane conformal (that is orientation and angle preserving) from the metric on X to the Euclidean metric on the w plane? This is the case if and only if the Jacobian $Jac(w) > 0$ and $|dz + \mu(z)d\bar{z}|^2$ is proportional to $|dw|$ everywhere.
The chain rule using the above notation becomes

$$dw = \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \bar{z}}d\bar{z} \tag{1.5}$$

and the Jacobian

$$Jac(w) = |w_z|^2 - |w_{\bar{z}}|^2. \tag{1.6}$$

From (1.5) we get that

$$|dw| = |w_z||dz + \frac{w_{\bar{z}}}{w_z}d\bar{z}| \tag{1.7}$$

and so by identifying $w_{\bar{z}}/w_z = \mu(z)$ the criterion for conformallity becomes

$$w_{\bar{z}} = \mu(z)w_z \qquad \text{at all points} \tag{1.8}$$

Definition 1.4.1. If w is an orientation-preserving homeomorphism of planar domains such that w is differentiable at z with $Jac(w) = |w_z|^2 - |w_{\bar{z}}|^2$ positive at z, then we define

$$\mu(z) = \frac{w_{\bar{z}}}{w_z}$$

to be the **complex dilatation** of w at z. Denote $\mu(z) = \mu_w(z)$, then $\mu_w(z)$ is a well-defined complex number of modulus less than 1 and at such points z is called a **regular point** for w.

The geometric interpretation of the complex dilatation is connected to the directional derivative of a mapping. In a sense the complex dilatation measures the distortion in the $\partial/\partial z$ and $\partial/\partial \bar{z}$ directions.
Let $\partial_\alpha f(z)$ denote the directional derivative of a $\mathcal{C}^1$ mapping $f(x, y)$ in a direction making an angle α with positive x-axis.
Then for w an orientation-preserving $\mathcal{C}^1$ map between planar domains

$$\max_\alpha |\partial_\alpha w(z)| = |w_z| + |w_{\bar{z}}|$$

$$\min_\alpha |\partial_\alpha w(z)| = |w_z| - |w_{\bar{z}}|$$

and the ration

$$\frac{\max_\alpha |\partial_\alpha w(z)|}{\min_\alpha |\partial_\alpha w(z)|} = \frac{1 + |\mu_w(z)|}{1 - |\mu_w(z)|}$$

depends only on $|\mu_w(z)|$.
By setting

$$L(z, r) = \max_\xi \{|w(\xi) - w(z)| : |\xi - z| = r\}$$

$$l(z, r) = \min_\xi \{|w(\xi) - w(z)| : |\xi - z| = r\}$$

for arbitrary homeomorphisms between planar domains, an orientation-preserving homeomorphism w from a planar domain D_1 onto a planar domain D_2 is **quasiconformal** if the circular dilatation at z

$$H(z) = \lim_{r \to 0} \frac{L(z, r)}{l(z, r)}$$

is globally bounded on D_1.
Then, for an orientation-preserving $\mathcal{C}^1$ diffeomorphism between planar domains, this is equivalent to

$$\sup_{z \in D_1} |\mu_w(z)| < 1.$$

Intuitively speaking, a homeomorphism is quasiconformal if and only if the amount of distortion it produces on infinitesimal circles is bounded as the center of the circles is moved around the domain.

Proposition 1.4.2. *For a quasiconformal homeomorphism w on D_1 almost every point of D_1 is a regular point. Thus, the complex dilatation $\mu_w(z)$ is defined almost everywhere on D_1.*

Quasiconformal homeomorphisms of planar domains form a pseudo group. This is because inverses and composites remain quasiconformal. Thus, we can talk about orientable surfaces with quasiconformal structure by assigning an atlas of charts whose transition functions are quasiconformal homeomorphisms. It then makes sense talking about locally quasiconformal maps between such surfaces. In particular a conformal atlas is quasiconformal. It also makes sense to speak about globally quasiconformal homeomorphisms between Riemann surfaces.

Given a function (Beltrami coefficient) μ in $L^\infty(D)_1$[13], the equation

$$w_{\bar{z}} = \mu(z) w_z$$

is called the Beltrami equation with coefficient μ. It is natural to call any quasiconformal map w satisfying the above equation a **μ-conformal homeomorphism of** D.
The following theorem shows that solutions to the Beltrami equation always exist.

Theorem 1.4.3. *Given any $\mu \in L^\infty(\mathbb{C})_1$, there exists a unique μ-conformal homeomorphism f from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ fixing $0, 1, \infty$.*

The unique quasiconformal homeomorphism f from the above theorem will be denoted w^μ.
The uniqueness of the theorem follows from the fact that the biholomorphic automorphisms of $\mathbb{C}$ are of the form $\sigma(z) = az + b$, $a \neq 0$, and so we can always normalize a μ-conformal map $f : \mathbb{C} \to \mathbb{C}$ by requiring that $f(0) = 0$ and $f(1) = 1$. The extension sending ∞ to ∞ is then quasiconformal from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. Thus, the normalization fixes f uniquely.
Now, the existence and uniqueness of theorem (1.4.3) guaranties that for arbitrary μ in $L^\infty(\mathbb{H})_1$ there is a quasiconformal homeomorphism of $\mathbb{H}$ onto itself with complex dilatation μ on $\mathbb{H}$.
This is of importance for deforming Fuchsian groups into new Fuchsian groups. In fact, given $\mu \in L^\infty(\mathbb{H})_1$, define $\mu^* \in L^\infty(\mathbb{C})_1$ by

$$\mu^*(z) = \begin{cases} \mu(z) & \text{for} \quad z \in \mathbb{H} \\ \\ \overline{\mu(\bar{z})} & \text{for} \quad z \in L \end{cases} \tag{1.9}$$

Then the normalized μ^*-conformal w^{μ^*} map guarantied by theorem (1.4.3) must map $\mathbb{H}$ to $\mathbb{H}$ and L to L ($\mathbb{H}$ and L denoting the upper and lower half-plane).

[13] $L^\infty(D)_1$ is the unit ball in the Banach space $L^\infty(D)$.

Thus, we denote w^{μ^*} (or rather its restriction to $\mathbb{H}$) w_μ and note that this is the unique μ-conformal self-homeomorphism of $\mathbb{H}$ whose extension to $\hat{\mathbb{R}}$ fixes $0, 1$ and ∞.
The quasiconformal homeomorphisms are precisely those that are obtained by holomorphically perturbing the identity homeomorphism, staying among the family of homeomorphism.
As we are interested in compact Riemann surfaces, the only groups that we will use are Fuchsian groups (of the first kind).

Quasiconformal mappings on Riemann surfaces

Let X be a Riemann surface. Now any $\mu \in L^\infty(\mathbb{H})_1$ assigns a complex analytic atlas on X whose local charts are quasiconformal with respect to the original complex structure of X. This new complex structure will be called the μ structure on X and X equipped with the new atlas is the Riemann surface X_μ.
The Riemann surface X_μ can be represented by finding a Fuchsian group Γ_μ where Γ is a Fuchsian group representing X (i.e. $X \cong \mathbb{H}/\Gamma$). The group Γ_μ is obtained as $w_\mu \Gamma w_\mu^{-1}$ where w_μ is a quasiconformal self-homeomorphism of $\mathbb{H}$.

Definition 1.4.4. We say that w **is compatible with** Γ if $w\Gamma w^{-1}$ is a group of Möbius transformations when ever Γ is any given Fuchsian group and w is any quasiconformal homeomorphism of $\widehat{\mathbb{C}}$.

Let $\Gamma \subset PSL_2(\mathbb{R})$ be a Fuchsian group acting on $\mathbb{H}$ and define the complex Banach space

$$L^\infty(\mathbb{H},\Gamma)_1 = \{\mu \in L^\infty(\mathbb{H})_1 : (\mu \circ g)\frac{\overline{g'}}{g'} = \mu \quad \text{a.e. in } \mathbb{H}, \quad \forall g \in G\} \quad (1.10)$$

consisting of Beltrami differentials for Γ on $\mathbb{H}$.
Note that for $\mu \in L^\infty(\mathbb{H},\Gamma)_1$, μ^* as defined on $\mathbb{C}$ in equation (1.9) automatically satisfies (1.10) on $\mathbb{C}$. Therefore, w_μ (the restriction of w^{μ^*} to $\mathbb{H}$) defined below equation (1.9) is compatible with Γ and also w^μ (which is μ-conformal in $\mathbb{H}$ and conformal in L) is compatible with Γ.

Proposition 1.4.5. *The element $g_\mu = w_\mu \circ g \circ w_\mu^{-1}$ of $PSL_2(\mathbb{R})$ depends real analytically on μ.*

If Γ is an arbitrary Fuchsian group and $g \in \Gamma$. Then $w \circ g \circ w^{-1}$ has fixed points precisely at the w image of the fixed points of g. Hence, since the parabolic, hyperbolic and elliptic type of a transformation in $PSL_2(\mathbb{R})$ is determined by its fixed points, g_μ is always the same type of Möbius transformation as g.

The metric of $L^\infty(\mathbb{H},\Gamma)_1$

Each $\mu \in L^\infty(\mathbb{H})_1$ determines a normalized μ-conformal automorphism w_μ of $\mathbb{H}$. The compositions and inverses of these elements provides a group structure on $L^\infty(\mathbb{H})_1$, where the multiplication is denoted by

$$\lambda \cdot \mu = \text{the complex dilatation of } w_\lambda \circ w_\mu$$

Proposition 1.4.6. *The unit ball $L^\infty(\mathbb{H})_1$ is a group under the above operations. Each right translation map R_θ, $R_\theta(\mu) = \mu \cdot \theta$ is a biholomorphic automorphism of $L^\infty(\mathbb{H})_1$. Thus, $L^\infty(\mathbb{H})_1$ is a holomorphically homogeneous complex Banach manifold.*

The real analytic elements of $L^\infty(\mathbb{H})_1$ form a subgroup. This subgroup corresponds to the group of normalized quasiconformal self-homeomorphisms of $\mathbb{H}$ which are real-analytic diffeomorphisms.
Now using the group structure of $L^\infty(\mathbb{H})_1$, we can put a complete metric on this unit ball, and therefore a complete metric on each $L^\infty(\mathbb{H},\Gamma)_1$ for arbitrary Fuchsian group.
We set

$$\widetilde{\tau}(\mu,\nu) = \frac{1}{2}\log\frac{1+\|\mu\cdot\nu^{-1}\|_\infty}{1-\|\mu\cdot\nu^{-1}\|_\infty}, \qquad \mu,\nu \in L^\infty(\mathbb{H})_1$$

where

$$\|\mu\cdot\nu^{-1}\|_\infty = \left\|\frac{\mu-\nu}{1-\mu\bar{\nu}}\right\|_\infty$$

It is not difficult to see that $\widetilde{\tau}$ is a metric on $L^\infty(\mathbb{H})_1$, and this metric is complete even when restricted to any $L^\infty(\mathbb{H},\Gamma)_1$.
Note that by the definition of the dilatation of a quasiconformal homeomorphism the above metric can also be written in terms of quasiconformal homeomorphisms

$$\widetilde{\tau}(\mu,\nu) = \frac{1}{2}\log K(w_\mu \circ w_\nu^{-1}) \tag{1.11}$$

where $K(w) = \frac{1+\|w\|_\infty}{1-\|w\|_\infty}$

The Teichmüller space of Riemann surfaces

Let X be a compact Riemann surface.

Definition 1.4.7. A **marked Riemann surface modelled on** X, is a triple (X, f, X_1), where X_1 is a Riemann surface and $f : X \to X_1$ is a quasiconformal homeomorphism. f is called the **marking map**.

Now, define $\hat{M}(X)$ by

$$\hat{M}(X) = \text{the set of all marked Riemann surfaces modelled on X.}$$

Definition 1.4.8. For the set $\hat{M}(X)$ we have:

1. (X, f, X_1) and (X, g, X_2) in $\hat{M}(X)$ are called **Teichmüller equivalent** ($\sim$) if and only if there is a biholomorphism $\sigma : X_1 \to X_2$ such that $g^{-1} \circ \sigma \circ f : X \to X$ is homotopic to the identity 1_X. The **Teichmüller space of** X, $T(X)$, is the set of equivalence classes $T(X) = \hat{M}(X)/\sim$.

2. (X, f, X_1) and (X, g, X_2) in $\hat{M}$ are called **Riemann equivalent** $\sim_R$ if and only if there exists a biholomorphism $\sigma : X_1 \to X_2$. The **Riemann moduli space of** X, $R(X)$, is the set of equivalence classes $R(X) = \hat{M}(X)/\sim_R$.

Using a diagram of homeomorphisms

$$\begin{tikzcd} & X_1 \arrow[dd, "\sigma"] \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & X_2 \end{tikzcd}$$

Figure 1.14: Diagram of homeomorphisms f, g and σ

we say that any such diagram **commutes up to homotopy** whenever $g^{-1} \circ \sigma \circ f$ is homotopic to 1_X

There is a natural projection between $T(x)$ and $R(X)$

$$T(X) \to R(X).$$

In the spaces above the equivalence class of (X, f, X_1) represents the biholomorphism class of the Riemann surface X_1. Thus a point in the Teichmüller space say $[X, f, X_1]$ represent the equivalence class of surfaces X_1 which look quasi-conformally like the reference surface X.

Our next aim is to topologize the above sets using some metric and the most natural way to do this is to use the Teichmüller distance defined in equation (1.11). Such a metric would be defined by

$$\hat{\tau} : \hat{M}(X) \times \hat{M}(X) \to \mathbb{R}^+$$

given by

$$\hat{\tau}((X, f, X_1), (X, g, X_2)) = \frac{1}{2} \log K(g \circ f^{-1}).$$

This is not a metric on $\hat{M}(X)$ since $\hat{\tau} = 0$ whenever $g \circ f^{-1}$ is conformal. This can be avoided by defining the **moduli space** $M(X) = \hat{M}(X)/\approx$ where the equivalence relation $\approx$ is given by

$$(X, f, X_1) \approx (X, g, X_2) \quad \text{if and only if } g \circ f^{-1} \text{ is conformal.}$$

If $(X, f, X_1) \approx (X, g, X_2)$ then obviously we can take σ to be $\sigma = g \circ f^{-1}$, and so the corresponding diagram commutes up to homotopy and thus, the two triplets are Teichmüller equivalent. Hence the quotient projection from $\hat{M}(X)$ to $T(X)$ factors through $M(X)$ and we get a sequence of natural projections

$$\hat{M}(X) \to M(X) \xrightarrow{\Phi} T(X) \to R(X) \tag{1.12}$$

We will denote the *fundamental projection* ϕ as follows

$$\Phi : M(X) \to T(X)$$

Now, $(M(X), \hat{\tau})$ is a complete metric space and via the natural projection Φ we get the quotient pseudometric[14] τ, induced on $T(X)$.
With this the definition of τ we get the following.

Definition 1.4.9. The **Teichmüller metric** τ on the Teichmüller space $T(X)$ is defined to be

$$\tau([X, f, X_1], [X, g, X_2]) = \inf_{\sigma}\{\frac{1}{2}\log K(\sigma)\} \tag{1.13}$$

where σ varies over all quasiconformal homeomorphisms $\sigma : X_1 \to X_2$ which are homotopic to $g \circ f^{-1}$.

Proposition 1.4.10. *(T, τ) is a complete connected metric space for any arbitrary Riemann surface X.*

The Teichmüller space of Fuchsian groups

Since any compact Riemann surface can be uniformized by a Fuchsian group Γ it is also natural to talk about the Teichmüller space of Γ, $T(\Gamma)$.
Let Γ be a Fuchsian group operating on $\mathbb{H}$, with or without elliptic elements. Recall the definition of the complex Banach space consisting of all Beltrami differentials for Γ on $\mathbb{H}$ given by

$$L^\infty(\mathbb{H}, \Gamma)_1 = \{\mu \in L^\infty(\mathbb{H})_1 : (\mu \circ g)\frac{\overline{g'}}{g'} = \mu \quad \text{a.e. in } \mathbb{H}, \quad \forall g \in \Gamma\}. \tag{1.14}$$

We have seen that a normalized quasiconformal automorphism of $\mathbb{H}$, w_μ, is compatible with Γ. This means that $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$ is again a discrete subgroup of $PSL_2(\mathbb{R})$ if and only if $\mu \in L^\infty(\mathbb{H}, \Gamma)_1$. Thus, each such μ determines a monomorphic[15] embedding E_μ of Γ into $PSL_2(\mathbb{R})$:

$$E_\mu : \Gamma \to \Gamma_\mu \quad \text{given by} \quad E_\mu(g) = g_\mu = w_\mu \circ g \circ w_\mu^{-1}.$$

Since conjugation by quasiconformal maps preserves the type of Möbius transformation, we know that the image group Γ_μ is also a Fuchsian group of the same signature as Γ.

[14] If $\Phi : (M, d) \to T$ is a surjection from a metric space (M, d) onto a set T, then the quotient pseudometric (via Φ) on T is defined to be $d_\Phi(t_1, t_2) = \inf\{d(m_1, M_2) : m_1 \in \Phi^{-1}(t_1), m_2 \in \Phi^{-1}(t_2)\}$. A pseudometric differs from a metric in the sense that the distance between two distinct points is allowed to be zero in a pseudometric.

[15] $\equiv$ injective.

Now, in order to parameterize all Riemann surfaces of the quasiconformal type $\mathbb{H}/\Gamma$ we need to parameterize their uniformizing groups Γ_μ. The Riemann moduli space will then be the obtained by identifying those μ and ν in $L^\infty(\mathbb{H},\Gamma)_1$ such that Γ_μ and Γ_ν are conjugate subgroups in $PSL_2(\mathbb{R})$.

Remark 1.4.11. *Recall theorem (1.3.32) which states that conjugating Γ with a conformal automorphism of $\mathbb{H}$ results in a new Fuchsian group, but this represents the same[16] Riemann surface.*

Thus, by constructing Fuchsian groups Γ_μ by conjugation by a quasiconformal automorphism w_μ of $\mathbb{H}$ we obtain quasi-conformally deformed Fuchsian groups that will represent exhaustively *all* possible structures in the Teichmüller space of $\mathbb{H}/\Gamma$.

Proposition 1.4.12. *Let Γ be a torsion-free Fuchsian group, and let $\mu, \nu \in L^\infty(\mathbb{H},\Gamma)_1$ be such that the image groups Γ_μ and Γ_ν coincide. Call $X = \mathbb{H}/\Gamma$, $Y = \mathbb{H}/\Gamma_\mu = \mathbb{H}/\Gamma_\nu$. Then w_μ and w_ν induce well-defined quasiconformal homeomorphisms, called f_μ and f_ν respectively, from X to Y. Then the following are equivalent:*

(i) f_μ is homotopic to f_ν.

(ii) $w_\mu|_{\hat{\mathbb{R}}} = w_\nu|_{\hat{\mathbb{R}})}$

(iii) The monomorphisms E_μ and E_ν of Γ into $PSL_2(\mathbb{R})$ coincide. That is $w_\mu \circ g \circ w_\mu^{-1} = w_\nu \circ g \circ w_\nu^{-1}$ for all $g \in \Gamma$.

Moreover, conditions (ii) and (iii) remains equivalent even in the case when Γ contains elliptic elements (i.e. torsion).

We state the interesting fact following the above proposition in a corollary

Corollary 1.4.13. *A diffeomorphism $f : X \to X$ is homotopic to the identity on a compact Riemann surface $X = \mathbb{H}/\Gamma$ if and only if there is a lift $\tilde{f}$ of f to the universal cover $\mathbb{H}$ such that $\tilde{f} : \mathbb{H} \to \mathbb{H}$ extends continuously by the identity map on the real axis.*

We can now define deformation spaces for arbitrary Fuchsian groups in the following way.

Definition 1.4.14. Let $\Gamma \subset PSL_2(\mathbb{R})$ be an arbitrary Fuchsian group.

1. The **Teichmüller space of** Γ, $T(\Gamma)$ (Γ non-elementary), is $L^\infty(\mathbb{H},\Gamma)_1/\sim$, where $\mu \sim \nu$ if and only if $\mu|_{\mathbb{R}} = \nu|_{\mathbb{R}}$ (That is if $E_\mu = E_\nu$). $T(\Gamma)$ is given the quotient topology from the Banach-norm topology on $L^\infty(\mathbb{H},\Gamma)_1$.

   The quotient pseudometric τ on $T(\Gamma)$ obtained from $\hat{\tau}$ is again a metric and τ also induces the quotient topology from $L^\infty(\mathbb{H},\Gamma)_1$.

[16] The same being biholomorphically equivalent in this context.

2. The **Riemann moduli space of** Γ, $R(\Gamma)$ is $L^\infty(\mathbb{H},\Gamma)_1/\sim_R$, where $\mu \sim_R \nu$ if and only if Γ_μ and Γ_ν are conjugate in $PSL_2(\mathbb{R})$.

As we have seen we also have the same sequence of projection mappings for these spaces.

$$L^\infty(\mathbb{H},\Gamma)_1 \stackrel{\Phi_\Gamma}{\to} T(\Gamma) \to R(\Gamma)$$

Now, $T(\Gamma)$ can be identified as the Teichmüller space of some Riemann surface (even when Γ has torsion) via a natural isomorphism ρ between $T(\Gamma)$ and $T(X)$, where X is the Riemann surface U_Γ/Γ and U_Γ is the open subset of $\mathbb{H}$ on which Γ acts without fixed points.
The main properties of the Teichmüller space $T(\Gamma)$ are given by the following theorem.

Theorem 1.4.15. *Let Γ be a Fuchsian group uniformizing a Riemann surface X_g of genus g.*

1. *$T(\Gamma)$ is a complete metric space of finite dimension $d(\Gamma)$.*

2. *If Γ has signature $s(\Gamma) = (g; m_1, \dots, m_r)$ then $T(\Gamma)$ is a cell of complex dimension $d(\Gamma) = 3g - 3 + r$.*[17]

3. *Given two Fuchsian groups Γ and Γ' and a group monomorphism $\alpha : \Gamma \to \Gamma'$ the induced map*

$$T(\alpha) : T(\Gamma) \to T(\Gamma')$$

defined by $[r] \mapsto [\alpha(r)]$, is an isometric embedding.[18]

The modular group

If we define

$$Q(\Gamma) = \{w \text{ quasiconformal homeomorphism of } \mathbb{H} : w\Gamma w^{-1} \subseteq PSL_2(\mathbb{R})\} \tag{1.15}$$

then the complex dilatation of $w \in Q(\Gamma)$ will be in $L^\infty(\mathbb{H},\Gamma)_1$, since $L^\infty(\mathbb{H},\Gamma)_1$ can be thought of as the normalized quasiconformal homeomorphisms of $\mathbb{H}$ which conjugate Γ to some other subgroup of $PSL_2(\mathbb{R})$. Hence there is a 1-1 correspondence between $L^\infty(\mathbb{H},\Gamma)_1$ and the group

$$Q_n(\Gamma) = \{w \in Q(\Gamma) : w \text{ fixes each of } 0, 1, \infty\} \tag{1.16}$$

Then by defining

$$Q_0(\Gamma) = \{w \in Q_n(\Gamma) : w|_\mathbb{R} = 1_\mathbb{R}\}, \tag{1.17}$$

we can now define the modular group of Γ.

[17]See [17] or theorem 2.5.5 in [33]. This theorem states that the Fricke mapping $F : T(X) \to \mathbb{R}^{6g-6+2r}$ is a real analytic embedding with real analytic inverse of the complex manifold T(X) onto an open domain $image(F) \subset \mathbb{R}^{6g-6+2r}$.
[18]See [19].

Definition 1.4.16. The **modular group of** Γ, $Mod(\Gamma)$ is defined to be the quotient

$$Mod(\Gamma) = Q(\Gamma)/Q_0(\Gamma).$$

Note that $Q_0(\Gamma)$ is a normal subgroup of $Q(\Gamma)$.
Again, we can also talk about the modular group of a Riemann surface. This is done by defining $Q(X)$ as the group of all orientation-preserving quasiconformal self-homeomorphisms of X and let $Q_0(X)$ be all $\phi \in Q(X)$ which are homotopic to the identity on X. Then Q_0 is a normal subgroup of $Q(X)$ and the quotient group

$$Mod(X) = Q(X)/Q_0(X)$$

is defined to be the modular group[19].
Since we are only dealing with compact Riemann surfaces, by Nielsen's theorem[20], $Mod(X)$ is canonically identifiable as

$$Mod(X) = \mathrm{Aut}^{+}\pi_1(X)/Inn(\mathrm{Aut}\,\pi_1(X))$$

where $\mathrm{Aut}^{+}\pi_1(X)$ is the index two subgroup in $\mathrm{Aut}\,\pi_1(X)$ consisting of all the orientation-preserving self-homeomorphisms. This coincides with the previous definition of the modular group.
Similarly for the modular group of a Fuchsian group Γ we can state that

$$Mod(\Gamma) = \mathrm{Aut}\,\Gamma/Inn(\Gamma)$$

where $Inn(\Gamma)$ is the normal subgroup of $\mathrm{Aut}\,\Gamma$ consisting of all inner automorphisms[21] of Γ.

Action of the modular group

Clearly, $T(X)$ does not depend on the reference Riemann surface X, but only on the quasiconformal type of X. Thus it makes sense of talking of $[X, 1, X] \in T(X)$ as a base point for $T(X)$.
Now if $\varphi : X \to Y$ is a quasiconformal homeomorphism, there is a natural map

$$\varphi^* : T(X) \to T(Y)$$

given by $\varphi^*([X, f, X_1]) = [Y, f \circ \varphi, X_1]$. Hence, φ^* maps $[X, \phi^{-1}, X_1]$ to the origin in $T(Y)$. Moreover, $[X, f, X_1]$ and its φ^* image represent the same Riemann surface (namely X_1) in $T(X)$ and $T(Y)$ respectively.

[19] The modular group is sometimes called the mapping class group or the quasiconformal mapping class group of X.
[20] (Nielsen). If X is a compact Riemann surface of positive genus, the homeomorphism $Homeo(X)/Homeo_0(X) \to \mathrm{Aut}\,\pi_1(X, x)/\mathrm{Aut}_0\pi_1(X, x)$ is an isomorphism.
[21] An inner automorphism of a group G is an element $\varphi \in \mathrm{Aut}\,G$ such that $\varphi(g) = hgh^{-1}$ for some fixed element $h \in G$.

One may think of the quasiconformal map φ as producing a change of base point for the Teichmüller space and so any point of $T(X)$ can serve as a base point.
Note that if $P_1 = [X, f, X_1]$ and $P_2 = [X, g, X_2]$ in $T(X)$ are such that there is a biholomorphism $\sigma : X_1 \to X_2$, then the modular action of $\varphi = g^{-1} \circ \sigma \circ f$ takes P_1 to $[X, \sigma \circ f, X_2] = [X, f, X_1] = P_1$. Conversely, if φ^* takes P_1 to P_2 then such a σ exists.
From this we get the following proposition.

Proposition 1.4.17. *The modular group $Mod(X)$ acts as a group of isometries on $T(X)$ and the action identifies precisely those points that represent biholomorphically equivalent Riemann surfaces. The quotient (orbit) space $T(X)/Mod(X)$ is canonically identified with the Riemann moduli space R(X).*

Maximal Fuchsian groups

Definition 1.4.18. A Fuchsian group Γ is called a **maximal Fuchsian group** if there is no other Fuchsian group Δ containing Γ with finite index and such that $d(\Gamma) = d(\Delta)$.

Likewise, a signature of a Fuchsian group is non-maximal if it is the signature of some non-maximal Fuchsian group.
The reason why maximal Fuchsian groups are of interest is the following proposition.

Proposition 1.4.19. *Given a compact Riemann surface $X = \mathbb{H}/\Gamma$, with Γ a Fuchsian surface group. Then the group $G = \Delta/\Gamma$ is the full group of automorphisms of X if and only if Δ is maximal.*

The importance of this proposition is that even if we know that a group of automorphisms G is given by Δ/Γ for some Fuchsian groups as defined above, we can not say whether it is the whole group of automorphisms or just a subgroup of the automorphism group until we know whether the Fuchsian group Δ is maximal.
Hence, we need to know whether a Fuchsian group is maximal or not and this can be done using the following theorem.

Theorem 1.4.20. *([19],[29]) The following conditions are equivalent:*

1. *$Mod(\Gamma)$ fails to act faithfully on $T(\Gamma)$.*

2. *There exists a Fuchsian group Γ' and a group monomorphism $\alpha : \Gamma \to \Gamma'$ such that $d(\Gamma) = d(\Gamma')$ and $\alpha(\Gamma)$ is a normal subgroup of Γ'.*

The full list of pairs of signatures $(s(\Gamma), s(\Gamma')$ of Fuchsian groups satisfying (2) in the above theorem was obtained by Singerman in [38].

$s(\Gamma)$	$s(\Gamma')$	$[\Gamma' : \alpha(\Gamma)]$
$(2; [-])$	$(0; [2, 2, 2, 2, 2, 2])$	2
$(1; [t, t])$	$(0; [2, 2, 2, 2, t])$	2
$(1; [t])$	$(0; [2, 2, 2, 2t])$	2
$(0; [t, t, u, u])$	$(0; [2, 2, t, u])$	2
$(0; [t, t, u])$	$(0; [2, t, 2u])$	2
$(0; [t, t, t, t])$	$(0; [2, 2, 2, t])$	4
$(0; [t, t, t])$	$(0; [3, 3, t])$	3
$(0; [t, t, t])$	$(0; [2, 3, 2t])$	6

Normal pairs of Fuchsian groups

To decide whether a given finite group can be the full group of automorphism of some compact Riemann surface we will need all pairs of signatures $s(\Gamma)$ and $s(\Gamma')$ for some Fuchsian groups Γ and Γ' such that $\Gamma \leq \Gamma'$ and $d(\Gamma) = d(\Gamma')$. The full list in the non-normal case was also obtained by Singerman in [38].

$s(\Gamma)$	$s(\Gamma')$	$[\Gamma' : \alpha(\Gamma)]$
$(0; [7, 7, 7])$	$(0; [2, 3, 7])$	24
$(0; [2, 7, 7])$	$(0; [2, 3, 7])$	9
$(0; [3, 3, 7])$	$(0; [2, 3, 7])$	8
$(0; [4, 8, 8])$	$(0; [2, 3, 8])$	12
$(0; [3, 8, 8])$	$(0; [2, 3, 8])$	10
$(0; [9, 9, 9])$	$(0; [2, 3, 9])$	12
$(0; [4, 4, 5])$	$(0; [2, 4, 5])$	6
$(0; [n, 4n, 4n])$	$(0; [2, 3, 4n])$	6
$(0; [n, 2n, 2n])$	$(0; [2, 4, 2n])$	4
$(0; [n, 3, 3n])$	$(0; [2, 3, 3n])$	4
$(0; [n, 2, 2n])$	$(0; [2, 3, 2n])$	3

Non-normal pairs of Fuchsian signatures

1.5 Equisymmetric Riemann surfaces and actions of groups

A finite subgroup of the modular group may be realized as a finite group of homeomorphisms of a surface. This is called the Nielsen realization problem and with this, modular group problems can sometimes be rephrased in terms of finite group actions on surfaces.

In the following we will present some background theory of equisymmetric Riemann surfaces and actions of finite groups on Riemann surfaces which can be found in Broughton's papers [8] and [9]. For a more general discussion see [6], [20] and [21].

Equisymmetric Riemann surfaces

We have seen that the moduli space of Riemann surfaces is given by the Teichmüller space under the action of the modular group. It consists of points representing conformally equivalent Riemann surfaces.
Furthermore, if $X = \mathbb{H}/\Gamma$ is a closed Riemann surface of genus g with $g \geq 2$, then $\mathrm{Aut}\,(X)$ determines a conjugacy class of subgroups $\Sigma(X)$ of $Mod(\Gamma)$. $\Sigma(X)$ is called the **symmetry type** of X. Clearly, two conformally equivalent Riemann surfaces have the same symmetry type. Hence one can talk about the symmetry type of the points of the moduli space.

Definition 1.5.1. Two closed Riemann surfaces are called **equisymmetric**, or are said to have the same symmetry type, if the two surfaces' conformal automorphism groups determine conjugate finite subgroups of the modular group. The subset of the moduli space corresponding to surfaces equisymmetric with given surface forms a locally closed subvariety of the moduli space, called an **equisymmetric stratum**.

We denote the equisymmetric strata by $\mathcal{M}_g^{|G|}$ where G is a subgroup of $Aut(X)$. The strata $\mathcal{M}_g^{|G|}$ is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$, the moduli group of Riemann surfaces of genus g.
The equisymmetric strata are in 1-1 correspondence with topological equivalence classes of orientation preserving actions of a finite group G on a Riemann surface X.
Two finite groups $G' \leq G$ can induce the same stratum if they are quotient groups of a pair of Fuchsian groups $\Delta' \leq \Delta$, where Δ' is a non-maximal Fuchsian group.

Finite group actions on Riemann surfaces

For an orientable surface X of genus g, let $Hom^+(X)$ be its group of orientation-preserving homeomorphisms and G a finite group.
Recall that a finite group G acts **effectively** and **orientably** on X if there is a monomorphism[22] $\varepsilon : G \to Hom^+(X)$.

Definition 1.5.2. Two effective and orientable actions defined by $\varepsilon : G \to Hom^+(X)$ and $\varepsilon' : G \to Hom^+(X)$ are said to be **topologically equivalent** if there is an $\omega \in \mathrm{Aut}\,(G)$ and an $h \in Hom^+(X)$ such that

$$\varepsilon'(g) = h\varepsilon(\omega(g))h^{-1}$$

for all $g \in G$.

Thus, two actions are equivalent if the following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{\varepsilon'} & Hom^+(X) \\ \omega \downarrow & & \downarrow \varphi \\ G & \xrightarrow{\varepsilon} & Hom^+(X) \end{array}$$

[22]An injective homomorphism

where φ is an inner automorphism of $Hom^+(X)$. In our discussion about compact Riemann surfaces $Hom^+(X)$ is simply $PSL_2(\mathbb{R})$ and then it is not hard to see that the two actions are topologically equivalent if they are conjugate elements in $PSL_2(\mathbb{R})$.

If X is uniformized by a Fuchsian surface group Γ such that $X = \mathbb{H}/\Gamma$, then each effective and orientable action of a finite group G on X is determined by an epimorphism

$$\theta : \Delta \to G$$

from the Fuchsian group Δ such that $ker(\theta) = \Gamma$.

Hence we obtain the following lemma.

Lemma 1.5.3. *Two epimorphisms $\theta_1 : \Delta \to G$ and $\theta_2 : \Delta \to G$ define two topologically equivalent actions of G on $X = \mathbb{H}/\Gamma$ if there exist two automorphisms $\phi : \Delta \to \Delta$ and $\omega : G \to G$ such that*

$$\theta_2 = \omega \circ \theta_1 \circ \phi^{-1}. \tag{1.18}$$

Proof. Let ε and ε' be two equivalent actions as above together with $\omega \in \mathrm{Aut}\,(G)$ and $h \in Hom^+(X)$ making the diagram commute. So there exist epimorphisms $\theta_1, \theta_2 : \Delta \to G$ determining the two actions.

Now the map h lifts to an orientation-preserving homeomorphism h^* of $\mathbb{H}$ such that $h^*\Delta(h^*)^{-1} = \Delta$. This induces the automorphism $\phi : \Delta \to \Delta$ given by

$$\phi(g) = h^* g (h^*)^{-1} \tag{1.19}$$

which makes the diagram commute

$$\begin{array}{ccc} \Delta & \xrightarrow{\theta_1} & G \\ \phi \downarrow & & \downarrow \omega \\ \Delta & \xrightarrow{\theta_2} & G \end{array}$$

giving us the desired $\theta_2 = \omega \circ \theta_1 \phi^{-1}$. □

Let $\mathcal{B}$ be the subgroup of $\mathrm{Aut}\,(\Delta)$ induced by orientation-preserving homeomorphisms as in (1.19). Then the group $\mathcal{B} \times \mathrm{Aut}\,(G)$ acts on the set of epimorphisms $\theta : \Delta \to G$ by the relation in (1.18). Thus, two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define the same equivalence class of G-actions if and only if they lie in the same $\mathcal{B} \times \mathrm{Aut}\,(G)$-class.

Algebraic characterization of $\mathcal{B}$

In the following we are only interested in Fuchsian groups with signature $(0; m_1, \ldots, m_1)$, that is Fuchsian groups uniformizing Riemann surfaces of genus 0.

In order to determine certain actions of a specific group we need to algebraically characterize $\mathcal{B} \subset \mathrm{Aut}\,(\Delta)$. We can construct the automorphisms of Δ by constructing them in a group lying over Δ.

If X/G is uniformized by a Fuchsian surface group Δ with signature $s(\Delta) = (0; m_1, \ldots, m_1)$ then the canonical representation of Δ is given by

$$\Delta = \langle x_1, \ldots, x_r | x_1^{m_1} = \ldots = x_r^{m_r} = \prod_{i=1}^{r} x_i = 1 \rangle \tag{1.20}$$

Then with F as the free group on the generators x_j define the group

$$\Pi = \langle x_1, \ldots, x_r | \prod_{i=1}^{r} x_i = 1 \rangle \tag{1.21}$$

Topologically, Π can be interpreted as the fundamental group of the punctured genus 0 surface S (punctured sphere), which is simply X/G with its r branch points removed. The elements of $\mathcal{B}$ can be identified with a certain subgroup of finite index of the modular group of S.

Proposition 1.5.4. *With Π as defined above, any automorphism of Δ can be lifted to an automorphism $\tilde{\phi}$ of Π such that*

1. *for each j, $\tilde{\phi}(x_j)$ is conjugate to some $\tilde{x}_j$.*

2. *The induced permutation representation $\mathcal{B} \to \Sigma_r$ preserves the branching orders.*

$$\begin{array}{ccc} \Pi & \xrightarrow{\tilde{\phi}} & \Pi \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\phi} & \Delta \end{array}$$

Now to prove that two actions are topologically equivalent, we need to find elements of $\mathcal{B} \times \mathrm{Aut}\,(G)$ such that the epimorphisms $\theta_1, \theta_2 : \Delta \to G$ are equivalent. The automorphisms can be produced ad hoc and are made of braid elements on the generators of Δ. These braid elements correspond to the automorphism $\phi_{i,i+1} \in \mathrm{Aut}\,(\Delta)$ defined by

1. $\phi_{i,i+1}(x_i) = x_i x_{i+1} x_i$,

2. $\phi_{i,i+1}(x_{i+1}) = x_i$,

3. $\phi_{i,i+1}(x_j) = x_j$ for all $j \neq i, i+1$.

Chapter 2

Trigonal Riemann surfaces of genus 4

This chapter is devoted to finding the different trigonal Riemann surfaces of genus 4 using their automorphism groups.
We begin with the study of cyclic trigonal Riemann surfaces.

2.1 Trigonal Riemann surfaces

Let X be a compact Riemann surface of genus $g \geq 2$ uniformized by a Fuchsian surface group Γ such that $X = \mathbb{H}/\Gamma$.

Definition 2.1.1. A Riemann surface X is called **trigonal** if it admits a three sheeted covering $f : X \to \widehat{\mathbb{C}}$ of the Riemann sphere. The covering f is called a **trigonal morphism**. If the trigonal morphism is a regular covering the covering is called a **cyclic trigonal morphism** and the Riemann surface is then called a **cyclic trigonal Riemann surface**.

The above definition is equivalent to the fact that X is represented as an algebraic curve[1] given by a polynomial equation of the form

$$y^3 + yb(x) + c(x) = 0.$$

If $b(x) \equiv 0$ then the trigonal morphism is a cyclic regular covering and the Riemann surface is called cyclic trigonal.
Using Fuchsian groups we are able to characterize the cyclic trigonal Riemann surfaces. This result is due to Costa and Izquierdo.

Theorem 2.1.2. *([12]) A Riemann surface X of genus g admits a cyclic trigonal morphism f if and only if there is a Fuchsian group Δ with signature*

$$s(\Delta) = (0; 3, 3, \overset{g+2}{\ldots}, 3) \tag{2.1}$$

[1]See [1], [3].

and an index three normal surface subgroup Γ *of* Δ*, such that* Γ *uniformizes* X*.*

Proof. Let X be a trigonal Riemann surface and f the trigonal morphism. Then there is an order 3 automorphism $\varphi : X \to X$ such that $X/<\varphi>$ is the Riemann sphere and φ is a deck-transformation of the covering map f. Let Γ be a Fuchsian surface group uniformizing X and let $\tilde{\varphi}$ be the lifting of φ to the universal covering $\mathbb{H} \to \mathbb{H}/\Gamma = X$. Then $\Delta =<\Gamma, \tilde{\varphi}>$ is the universal covering transformations group of X. Since $f : \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$ is a three sheeted regular covering, using the Riemann-Hurwitz formula (see corollary 1.3.7) we obtain

$$2g-2 = 3(2\cdot 0 - 2) + \sum_{i=1}^{r}(1-\frac{1}{3}). \tag{2.2}$$

From this we obtain that the number of branch point of order 3 is $r = g+2$ and so the signature of Δ is given by $s(\Delta) = (0; 3, 3, \overset{g+2}{\ldots}, 3)$ and Γ is an index three normal surface subgroup of Δ.
Conversely, if Δ is a Fuchsian group with signature $s(\Delta) = (0; 3, 3, \overset{g+2}{\ldots}, 3)$ and Γ an index three normal surface subgroup of Δ, such that $X = \mathbb{H}/\Gamma$. Then $f : X = \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$ is a cyclic trigonal morphism and X is a cyclic trigonal Riemann surface. □

A very useful result of Accola will also tell us about the number of trigonal morphisms.

Lemma 2.1.3. *([1]) Suppose* $X_g \to X_q$ *is a p-sheeted covering.*

(i) If p *is prime and* $g > 2pq + (p-1)^2$ *then there is only one such cover with the given* p *and* q*.*

(ii) If $X_g \to X_{q'}$ *is an* p'*-sheeted covering where* $q' = 0$*,* $p' = 3$ *and* $p = 2$ *then* $g \leq 2q+2$*.*

(iii) If $p = 2$ *and* $g > 4q+1$ *then there is a unique automorphism* T *of period two such that the genus of* $X_g/<T>$ *is two.*

Hence, for **trigonal Riemann surfaces of genus** $g \geq 5$ **the trigonal morphism is unique**. In this case the trigonal morphism is induced by a normal subgroup C_3 in $Aut(X_g)$ (This is a result of González-Díez [18]). Another useful result that we will use is the following.

Theorem 2.1.4. *([18]) If the automorphism group of a Riemann surface* X*,* $Aut(X)$*, contains automorphisms* τ_1 *and* τ_2 *of the same prime order and such that the quotient surfaces* X/τ_i*,* $(i = 1, 2)$*, are isomorphic to* $\widehat{\mathbb{C}}$*. Then* $<\tau_1>$ *and* $<\tau_2>$ *are conjugate in* $Aut(X)$*.*

In terms of cyclic trigonal Riemann surfaces, this tells us that the trigonal morphism f is unique if and only if $< \varphi >$ is a normal subgroup of G.
Our interest lies mainly in cyclic trigonal Riemann surfaces with non-unique trigonal morphisms. Therefore, we restrict ourself to study trigonal Riemann surfaces of genus 4.

2.2 Existence of cyclic trigonal Riemann surfaces of genus 4

The following results appear in

[13] A. Costa, M. Izquierdo, D. Ying, *On Riemann surfaces with non-unique cyclic trigonal morphism*, Manuscripta Math **118**, (2005) 443-453.

[22] M. Izquierdo, D. Ying, *On the space of Cyclic trigonal Riemann surfaces of genus 4*, To appear in Groups St. Andrews 2005, London Math. Soc. Lecture Notes, Cambridge University Press, (2006).

In order to find trigonal Riemann surfaces of genus 4 we use the theorems (1.3.31) and (2.1.2) in order to find all signatures of Fuchsian groups uniformizing the quotient surface X_4/G. The method is described in the following algorithm.

Producing cyclic trigonal Riemann surfaces of genus 4

Algorithm

Let X_4 be a Riemann surface of genus 4 uniformized by a Fuchsian surface group Γ. Furthermore, let G be its full group of automorphisms. Then X_4 admits a cyclic trigonal morphism f if and only if

1. there is a maximal Fuchsian group Δ with signature $(0; m_1, \ldots, m_r)$,
2. a trigonal automorphism $\varphi : X_4 \to X_4$ such that $< \varphi > \leq G$ and
3. an epimorphism $\theta : \Delta \to G$ such that $\theta^{-1}(< \varphi >)$ is a Fuchsian group with signature $(0; 3, 3, 3, 3, 3, 3)$.

$$
\begin{tikzcd}
X_4 = \mathbb{H}/\Gamma \arrow[dr] \arrow[dd] & \\
& \mathbb{H}/\Lambda = X_4/ < \varphi > \quad s(\Lambda) = (0; 3, 3, 3, 3, 3, 3) \arrow[dl] \\
X_4/G = \mathbb{H}/\Delta &
\end{tikzcd}
$$

Figure 2.1: Schematic diagram over the setting.

Assuming we have a cyclic trigonal surface, then the above epimorphism $\theta : \Delta \to G$ exists and $s(\Delta) = (0; m_{1_N}, \ldots, m_{r_N})$, where the integers m_{i_N} runs over the divisors of $|G|$.
Suppose that $|G| = N$ [2] applying the Riemann-Hurwitz formula from theorem (1.3.31) we get

$$\frac{\mu(\Lambda)}{\mu(\Delta)} = [\Delta : \Lambda] = \frac{|G|}{| < \varphi > |} = \frac{|G|}{3} \tag{2.3}$$

and so

$$3\mu(\Lambda) = |G|\mu(\Delta). \tag{2.4}$$

By the area formula for Fuchsian groups (equation 1.3) this now becomes

$$2(g + N - 1) = N \sum_{i_N=1}^{r_N} \frac{m_{i_N} - 1}{m_{i_N}}. \tag{2.5}$$

Solving this equation and using the list of maximal signatures ([38]) gives the following list of allowed signatures.

Lemma 2.2.1. *Let $X_4 = \mathbb{H}/\Gamma$ be a Riemann surface of genus 4, uniformized by a Fuchsian surface group Γ, admitting a cyclic trigonal morphism $f : X_4 \to \widehat{\mathbb{C}}$. Then the Fuchsian group uniformizing the orbifold X/G must have one of the following signatures.*

$\lvert G\rvert$	$s(\Delta)$	$\lvert G\rvert$	$s(\Delta)$	$\lvert G\rvert$	$s(\Delta)$
3	$(0; 3, 3, 3, 3, 3, 3)$	*6*	$(0; 2, 6, 6, 6)$	*6*	$(0; 2, 2, 3, 3, 3)$
6	$(0; 2, 2, 2, 3, 6)$	*6*	$(0; 2, 2, 2, 2, 2, 2)$	*6*	$(0; 3, 3, 6, 6)^*$
9	$(0; 9, 9, 9)^*$	*9*	$(0; 3, 3, 3, 3)^*$	*12*	$(0; 4, 6, 12)$
12	$(0; 2, 2, 2, 2, 2)$	*12*	$(0; 2, 3, 3, 3)$	*12*	$(0; 2, 2, 3, 6)$
12	$(0; 6, 6, 6)^*$	*12*	$(0; 3, 12, 12)^*$	*12*	$(0; 2, 2, 4, 4)^*$
15	$(0; 5, 5, 5)^*$	*15*	$(0; 3, 5, 15)^*$	*18*	$(0; 2, 2, 2, 6)$
18	$(0; 3, 6, 6)^*$	*18*	$(0; 2, 9, 18)^*$	*18*	$(0; 2, 2, 3, 3)^*$
21	$(0; 3, 3, 21)$	*24*	$(0; 3, 4, 6)$	*24*	$(0; 2, 2, 2, 4)$
24	$(0; 4, 4, 4)^*$	*24*	$(0; 3, 3, 12)^*$	*24*	$(0; 2, 8, 8)^*$
24	$(0; 2, 6, 12)^*$	*27*	$(0; 3, 3, 9)^*$	*30*	$(0; 2, 5, 10)^*$
36	$(0; 2, 4, 12)$	*36*	$(0; 2, 2, 2, 3)$	*36*	$(0; 3, 3, 6)^*$
36	$(0; 3, 4, 4)^*$	*36*	$(0; 2, 6, 6)^*$	*42*	$(0; 2, 3, 42)$
45	$(0; 3, 3, 5)^*$	*48*	$(0; 2, 3, 24)$	*48*	$(0; 2, 4, 8)^*$
54	$(0; 2, 3, 18)$	*60*	$(0; 2, 3, 15)$	*60*	$(0; 2, 5, 5)^*$
72	$(0; 2, 4, 6)$	*72*	$(0; 2, 3, 12)$	*72*	$(0; 3, 3, 4)^*$
90	$(0; 2, 3, 10)$	*108*	$(0; 2, 3, 9)$	*120*	$(0; 2, 4, 5)$
144	$(0; 2, 3, 8)$	*252*	$(0; 2, 3, 7)$		

** non-maximal signature*

[2]Recall that $N \leq 84(g - 1) = 252$.

Theorem 2.2.2. *There are Riemann surfaces of genus four admitting several cyclic trigonal morphisms. The cyclic trigonal Riemann surfaces of genus 4, classified according to the order of their automorphism group, are given by the following list.*

Riemann surface	*Order of aut. grp.*	*Determined by Fuchsian groups with signature*	*Automorphism group*
$V_4(\lambda, \delta, \gamma)$	3	$(0; 3, 3, 3, 3, 3, 3)$	C_3
$S_4(\lambda, \delta)$	6	$(0; 2, 2, 3, 3, 3)$	D_3 *or* C_6
$R_4(\lambda)$	12	$(0; 2, 2, 3, 6)$	D_6 *or* $C_6 \times C_2$
T_4	15	$(0; 3, 5, 15)$	C_{15}
$U_4(\lambda)$	18	$(0; 2, 2, 3, 3)$	$D_3 \times C_3$
$X(\lambda)$	36	$(0; 2, 2, 2, 3)$	$D_3 \times D_3$
Z_4	36	$(0; 2, 6, 6)$	$C_6 \times D_3$
X_4	72	$(0; 2, 3, 12)$	$\Sigma_4 \times C_3$
Y_4	72	$(0; 2, 4, 6)$	$(C_3 \times C_3) \rtimes D_4$

The names of the Riemann surfaces above are given arbitrary and the parameters given, λ, δ and γ, are depending on the dimension of the space of trigonal Riemann surfaces in each case. These spaces will be more rigorously investigated in the following sections. For now notice that in general these spaces of Riemann surfaces will be denoted ${}^{|G|}\mathcal{M}_g^p$ where $|G|$ is the order of the automorphism group, p is the order of the morphism and g is the genus of the surfaces.

Proof. We use theorem (1.3.31) to calculate $\theta^{-1}(\langle\varphi\rangle)$. The Fuchsian groups Δ have signatures as in lemma (2.2.1) and each case is separated according to the order of G.
$|G| = 3$.

There are epimorphisms $\theta : \Delta \to C_3 = \langle a | a^3 = 1 \rangle$ where $s(\Delta) = (0; 3, 3, 3, 3, 3, 3)$. For instance the epimorphisms,

$$\theta_1 \begin{cases} \theta_1(x_{2i}) = a \\ \theta_1(x_{2i-1}) = a^{-1} \\ 1 \leq i \leq 3 \end{cases}$$

and

$$\theta_2 \begin{cases} \theta_2(x_i) = a \\ 1 \leq i \leq 6 \end{cases}$$

Notice that $\theta^{-1}(C_3)$ has six cone points, and so we get that $s(\theta^{-1}(C_3)) = (0; 3, 3, 3, 3, 3, 3)$, giving the desired cyclic trigonal Riemann surfaces.

$|G| = 6$.

(i) First, consider the signature $s(\Delta_1) = (0; 2, 6, 6, 6)$. There are epimorphisms $\theta : \Delta \to C_6 = \langle a | a^6 = 1 \rangle$. The cone points in $\mathbb{H}/Ker(\theta)$ are given by the action of $\theta(x_2)$, $\theta(x_3)$ and $\theta(x_4)$ on the $\langle a^2\rangle$-cosets. Each such element gives one cone point of order 3. In total there are 3 cone points of order 3 and using equation (2.4) we get the equation

$$3(2g-2+3\cdot(1-\frac{1}{3})) = 6(2\cdot 0-2+\frac{1}{2}+3(1-\frac{1}{6})), \tag{2.6}$$

which gives us that the underlying surface has genus $g=1$ and so the group $\theta^{-1}(\langle a^2\rangle)$ has signature $(1;3,3,3)$ and the surfaces $\mathbb{H}/Ker(\theta)$ are not trigonal.

(ii) Consider now the signature $s(\Delta_2) = (0;2,2,3,3,3)$. There are possible epimorphisms from Δ_2 onto both D_3[3] and C_6[4].

Both C_6 and D_3 contain a unique normal subgroup of order 3, and they are generated by the elements b in C_6 and a in D_3. So there are 2 cosets in each group.

	$\langle\tau = b^2\rangle$		$\langle\tau = a\rangle$
$[1]$	$\{1, b^2, b^4\}$	$[1]$	$\{1, a, a^2\}$
$[b]$	$\{b, b^3, b^5\}$	$[s]$	$\{s, sa, sa^2\}$

For the epimorphism $\theta : \Delta_2 \to C_6$ there are several ways of choosing the epimorphism. On way is to take

$$\theta : \begin{cases} \theta(x_1) = b^3 \\ \theta(x_2) = b^3 \\ \theta(x_3) = b^2 \\ \theta(x_4) = b^2 \\ \theta(x_4) = b^2 \end{cases}$$

where $\theta(x_1x_2x_3x_4x_5) = 1_d$ and $\theta(\Delta_3) = C_6$.

The action of each element $\theta(x_3)$, $\theta(x_4)$ and $\theta(x_5)$ on the $\langle b^2\rangle$-cosets give the following orbits

$$\{[1]\}, \quad \{[b]\}$$

and thus producing two cone points each in $\mathbb{H}/Ker(\theta)$ giving a total of six cone points of order 3. Hence the signature of $\theta^{-1}(\langle b^2\rangle)$ is $(0;3,3,3,3,3,3)$ and the surfaces are trigonal with unique trigonal morphisms.

Similarly, for the epimorphism $\theta : \Delta_2 \to D_3$ there are several ways of choosing the epimorphism. On way is to take

[3] $D_3 = \langle a, s | a^3 = s^2 = (st)^2 = 1\rangle$
[4] $C_6 = \langle b | b^6 = 1\rangle$

$$\theta' \begin{cases} \theta'(x_1) = s \\ \theta'(x_2) = s \\ \theta'(x_3) = a \\ \theta'(x_4) = a \\ \theta'(x_5) = a \end{cases}$$

where $\theta'(x_1x_2x_3x_4x_5) = 1_d$ and $\theta'(\Delta_2) = D_3$.

The action on the $\langle a\rangle$-cosets, by the elements $\theta'(x_3)$, $\theta'(x_4)$ and $\theta'(x_5)$ give the orbits

$$\{[1]\}, \quad \{[s]\}$$

where again, each produces two cone points of order 3 giving a total of six order 3 cone points in $\mathbb{H}/Ker(\theta')$ and the surfaces are trigonal with unique trigonal morphisms.

(iii) Consider the signature $s(\Delta_3) = (0; 2, 2, 2, 3, 6)$. There are epimorphisms $\theta : \Delta_3 \to C_6$. $\theta(x_4)$ induces two cone points, while $\theta(x_5)$ induces one cone point in $\mathbb{H}/Ker(\theta)$. These surfaces are not trigonal since we again can use equation (2.4) to calculate the genus of the underlying surface and obtain that $s(\theta^{-1}(\langle a^2\rangle)) = (1; 3, 3, 3)$.

(iv) The signature $s(\Delta_4) = (0; 2, 2, 2, 2, 2, 2)$ does not produce any trigonal surface since the orders of the elliptic elements of Δ_4 are relative prime to 3.

(v) Fuchsian groups with signature $s(\Delta_5) = (0; 3, 3, 6, 6)$ are non-maximal. The epimorphisms $\overline{\theta} : \Delta_5 \to C_6$ lift to epimorphisms $\theta : \Delta \to C_6 \times C_2$ and $\theta : \Delta \to D_6$, where the signature of Δ is $s(\Delta) = (0; 2, 2, 3, 6)$. This signature is examined in the case where $|G| = 12$ (iii).

$|G| = 9$.

(i) Fuchsian groups with signature $s(\Delta_1) = (0; 9, 9, 9)$ are non-maximal. The epimorphisms $\overline{\theta} : \Delta_1 \to C_9$ lift to epimorphisms $\theta : \Delta \to C_{18}$, where $s(\Delta) = (0; 2, 9, 18)$.

(ii) Fuchsian groups with signature $s(\Delta_2) = (0; 3, 3, 3, 3)$ are non-maximal. The epimorphisms $\overline{\theta} : \Delta_2 \to C_3 \times C_3$ lift to epimorphism $\theta : \Delta \to C_{18}$, where $s(\Delta) = (0; 2, 2, 3, 3)$.

The Fuchsian groups with signature $(0; 2, 9, 18)$ and $(0; 2, 2, 3, 3)$ and their corresponding epimorphisms will be studied in the case when $|G| = 18$.

$|G| = 12$.

(i) First, consider the signature $s(\Delta_1) = (0; 4, 6, 12)$. There are epimorphisms $\theta : \Delta \to C_{12} = \langle a | a^{12} = 1\rangle$. The cone points in $\mathbb{H}/Ker(\theta)$ are given by the action of $\theta(x_2)$ and $\theta(x_3)$ on the $\langle a^4\rangle$-cosets. $\theta(x_2)$ induces two cone points, while $\theta(x_3)$ induces one cone point in $\mathbb{H}/Ker(\theta)$.

Thus, the signature (again using equation (2.4)) is $s(\theta^{-1}(\langle a^4\rangle)) = (1;3,3,3)$ and the surface $\mathbb{H}/Ker(\theta)$ is not trigonal.

(ii) Consider the signature $s(\Delta_2) = (0;2,3,3,3)$. The only group of order 12 generated by three elements of order 3 is A_4[5] (see [16]). This group contains one class of elements of order 3 represented by a since all elements of order 3 are conjugated. The respective coset table is given by

	$\langle\tau = a\rangle$
$[1]$	$\{1, a, a^2\}$
$[s]$	$\{s, sa, sa^2\}$
$[asa^2]$	$\{asa^2, as, asa\}$
$[a^2sa]$	$\{a^2sa, sas, a^2s\}$

There are epimorphisms $\theta : \Delta_2 \to A_4$. For instance

$$\theta : \begin{cases} \theta(x_1) = s \\ \theta(x_2) = a \\ \theta(x_3) = as \\ \theta(x_4) = sas \end{cases}$$

Consider the action of $\theta(x_3)$ on the $\langle a\rangle$-cosets. This has the orbits

$$\{[1]\}, \quad \{[s], [asa^2], [a^2sa]\},$$

leaving only one fixed coset.

Hence, each action yields only one cone point of order 3 and by equation (2.4) we get that $\theta^{-1}(C_3)$ has signature $(1;3,3,3)$ and the corresponding surfaces are not trigonal.

(iii) Now, consider the signature $s(\Delta_3) = (0;2,2,3,6)$. There are possible epimorphisms from Δ_3 onto both D_6[6] and $C_6 \times C_2$[7].

Both these groups contain unique normal subgroups of order 3. They are generated by the elements a^2 in $C_6 \times C_2$ and a^2 in D_6. Each group contains 4 cosets given by

	$\langle\tau = a^2\rangle$
$[1]$	$\{1, a^2, a^4\}$
$[a]$	$\{a, a^3, a^5\}$
$[s]$	$\{s, sa^2, sa^4\}$
$[sa]$	$\{sa, sa^3, sa^5\}$

[5] $A_4 = \langle a, s | a^3 = s^2 = (as)^3 = 1\rangle$
[6] $D_6 = \langle a, s | a^6 = s^2 = (as)^2\rangle$=1
[7] $C_6 \times C_2 = \langle a, s | a^6 = s^2 = [a, s] = 1\rangle$

Now the epimorphism $\theta : \Delta_3 \to C_6 \times C_2$ is given, for example by,

$$\theta : \begin{cases} \theta(x_1) = s \\ \theta(x_2) = sa^3 \\ \theta(x_3) = a^2 \\ \theta(x_4) = a \end{cases}$$

Consider the action of this epimorphism on the $\langle a^2 \rangle$-cosets. The elements $\theta(x_1)$ and $\theta(x_2)$ does not give any cone points. The elements of order 3, $\theta(x_3)$, gives the orbits

$$\{[1]\}, \quad \{[a]\}, \quad \{[s]\}, \quad \{[sa]\}.$$

and so they produce four cone points of order 3. The elements of order 6, $\theta(x_4)$ give the orbits

$$\{[1], [a]\}, \quad \{[s], [sa]\}$$

and so they produce two cone points of order 3.

In total there are six cone points of order 3 in $\mathbb{H}/Ker(\theta)$, giving $s(\theta^{-1}(\langle a^2 \rangle)) = (0; 3, 3, 3, 3, 3, 3)$ and the surfaces are trigonal with unique trigonal morphisms.

Similarly, the epimorphism $\theta' : \Delta_3 \to D_6$ given by

$$\theta' : \begin{cases} \theta'(x_1) = s \\ \theta'(x_2) = sa^3 \\ \theta'(x_3) = a^2 \\ \theta'(x_4) = a \end{cases}$$

In this case the orbits of the action of $\theta'(x_3)$ and $\theta'(x_4)$ are as the epimorphism θ above. Hence this action also produces six cone points of order 3 in $\mathbb{H}/Ker(\theta')$ and the surfaces are trigonal with unique trigonal morphisms.

(iv) The signature $s(\Delta_4) = (0; 2, 2, 2, 2, 2)$ does not produce any trigonal surface since the orders of the elliptic elements of Δ_4 are relative prime to 3.

(v) Fuchsian groups with signature $s(\Delta_5) = (0; 6, 6, 6)$ are non-maximal. The epimorphisms $\overline{\theta} : \Delta_5 \to C_6 \times C_2$ lift to epimorphisms $\theta : \Delta \to C_3 \times D_4$, where $s(\Delta) = (0; 2, 6, 12)$, and the epimorphism $\theta' : \Lambda \to C_3 \times A_4$, with $s(\Lambda) = (0; 3, 3, 6)$.

These groups appear in the cases when $|G| = 24$ and $|G| = 36$, respectively, but none of them are maximal. Therefore, as we shall see, in their respective cases they will again just lift to higher cases. Finally, they will end up in the case where $|G| = 72$ (ii).

(vi) Groups with signature $s(\Delta_6) = (0; 3, 12, 12)$ are non-maximal. The epimorphisms $\overline{\theta} : \Delta_6 \to C_{12}$ lift to epimorphisms $\theta : \Delta \to D_4 \times C_3$. This will be studied in the case when $|G| = 24$.

(vii) The signature $S(\Delta_7) = (0; 2, 2, 4, 4)$ can not yield trigonal Riemann surfaces since the orders of the elliptic generators of Δ_7 are relative prime to 3.

$|G| = 15$.

(i) First, consider the signature $s(\Delta_1) = (0; 5, 5, 5)$. There is no such epimorphism, since the only possible epimorphism from Δ_1 should be onto C_{15}, but C_{15} is not generated by elements of order 5.

(ii) Next, consider the signature $s(\Delta_2) = (0; 3, 5, 15)$. There are possible epimorphisms[8], for example $\theta : \Delta_2 \to C_{15}$ defined by

$$\theta : \begin{cases} x_1 \mapsto a^5 \\ x_2 \mapsto a^6 \\ x_3 \mapsto a^4 \end{cases}$$

The group C_{15} contains one normal subgroup of order 3, namely $\langle a^5 \rangle$. Its cosets are the following

	$\langle \tau = a^5 \rangle$
$[1]$	$\{1, a^5, a^{10}\}$
$[a]$	$\{a, a^6, a^{11}\}$
$[a^2]$	$\{a^2, a^7, a^{12}\}$
$[a^3]$	$\{a^3, a^8, a^{13}\}$
$[a^4]$	$\{a^4, a^9, a^{14}\}$

The action of $\theta(x_1)$ on the $\langle a^5 \rangle$-cosets gives the following orbits

$$\{[1]\}, \quad \{[a]\}, \quad \{[a^2]\}, \quad \{[a^3]\}, \quad \{[a^4]\},$$

leaving five cosets fixed, which yields 5 cone points of order 3.

The action of $\theta(x_2)$ on the $\langle a^5 \rangle$-cosets gives the orbit

$$\{[1], [a], [a^2], [a^3], [a^4]\}$$

of length 5 and hence does not produce any cone points.

Finally, $\theta(x_3)$ acts on the $\langle a^5 \rangle$-cosets and produces an orbit of length 5,

$$\{[1], [a^4], [a^3], [a^2], [a^1]\},$$

but since the order of the element is 15, this leaves one cone point of order 3.

[8]Later we will study this class more extensively.

In total there are six cone points of order 3 in $\mathbb{H}/Ker(\theta)$, giving $s(\theta^{-1}(\langle a^5\rangle)) = (0; 3, 3, 3, 3, 3, 3)$. Hence the surfaces are trigonal with unique trigonal morphism.

$|G| = 18$

(i) Consider the signature $s(\Delta_1) = (0; 2, 2, 2, 6)$. There is no group of order 18 generated by three involutions and containing elements of order 6 (See [16]). Hence there are no epimorphisms θ.

(ii) Now, consider the signature $s(\Delta_2) = (0; 2, 9, 18)$. The epimorphisms $\theta : \Delta_2 \to C_{18} = \langle a | a^{18} = 1\rangle$ are defined as $\theta(x_1) = a^9$ and $\theta(x_2) = a^{\pm 2i}$, $i = 1, 2, 4$. When acting on the $\langle a^6\rangle$-cosets, any element of order 9 in C_{18} leaves two cosets fixed and the elements of order 18 leaves one coset fixed. Hence $s(\theta^{-1}(C_3) = (1; 3, 3, 3)$ and the corresponding surfaces are not trigonal.

(iii) Groups with signature $s(\Delta_3) = (0; 3, 6, 6)$ are non-maximal. The epimorphisms $\overline{\theta}_1 : \Delta_3 \to C_6 \times C_3$ [9], defined as $\overline{\theta}_1(x_1) = b$, $\overline{\theta}_1(x_2) = tab$, and $\overline{\theta}_2 : \Delta_3 \to C_3 \times D_3$ [10], defined as $\overline{\theta}_2(x_1) = b$, $\overline{\theta}_2(x_2) = sab$, lift to an epimorphism $\theta : \Delta \to C_6 \times D_3$, where $s(\Delta) = (0; 2, 6, 6)$.

This epimorphism will be studied in the case when $|G| = 36$.

There is also one epimorphism $\overline{\theta}_3 : \Delta_3 \to C_3 \times D_3$ defined as $\overline{\theta}_3(x_1) = a^2b$, $\overline{\theta}_3(x_2) = sab$, which lifts to an epimorphism $\overline{\phi}_2 : \Delta \to D_3 \times D_3$, with $s(\Delta) = (0; 2, 6, 6)$. This last epimorphism lifts to an epimorphism that will be studied in the case when $|G| = 72$ (case (ii)).

(iv) Groups with signature $s(\Delta_4) = (0; 2, 2, 3, 3)$ are extensions of groups with signature $(0; 3, 3, 3, 3)$. Consider epimorphisms $\theta_{1,3} : \Delta_4 \to C_3 \times D_3$ and $\theta_2 : \Delta_4 \to \langle 3, 3, 3, 2\rangle = (C_3 \times C_3) \rtimes C_2$[11] defined as follows:

$$\theta_1 : \Delta_4 \to C_3 \times D_3 \begin{cases} \theta_1(x_1) = s \\ \theta_1(x_2) = sa \\ \theta_1(x_3) = ab \\ \theta_1(x_4) = ab^2 \end{cases}$$

$$\theta_2 : \Delta_4 \to \langle 3, 3, 3, 2\rangle \begin{cases} \theta_2(x_1) = sa \\ \theta_2(x_2) = sb \\ \theta_2(x_3) = ab \\ \theta_2(x_4) = b^2 \end{cases}$$

$$\theta_3 : \Delta_4 \to C_3 \times D_3 \begin{cases} \theta_3(x_1) = sa \\ \theta_3(x_2) = sa^2 \\ \theta_3(x_3) = a^2b \\ \theta_3(x_4) = b^2 \end{cases}$$

[9] $C_6 \times C_3 = \langle a, b, t | a^3 = b^3 = t^2 = [a, b] = [a, t] = [t, b] = 1\rangle$
[10] $C_3 \times D_3 = \langle a, b, s | a^3 = b^3 = s^2 = [a, b] = (sa)^2 = [s, b] = 1\rangle$
[11] $\langle 3, 3, 3, 2\rangle = \langle a, b, s | a^3 = b^3 = s^2 = (sa)^2 = (sb^2) = [a, b] = 1\rangle$

The epimorphisms θ_1 and θ_2 lift to epimorphisms $\theta_: \Delta \to D_3 \times D_3$, with $s(\Delta) = (0; 2, 2, 2, 3)$ which will be studied in the case when $|G| = 36$.

The last epimorphisms, $\theta_3 : \Delta_4 \to C_3 \times D_3$ yields maximal Fuchsian groups Δ_4. There are three classes of subgroups[12], represented by $\langle a \rangle$, $\langle b \rangle$ and $\langle ab \rangle$, of order 3 in $C_3 \times D_3$ and their respective cosets are given by

$\langle \tau_1 = a \rangle$	$\langle \tau_2 = b \rangle$	$\langle \tau_3 = ab \rangle$
$\{1, a, a^2\}$	$\{1, b, b^2\}$	$\{1, ab, a^2b^2\}$
$\{b, ab, a^2b\}$	$\{a, ab, ab^2\}$	$\{a, a^2b, b^2\}$
$\{b^2, ab^2, a^2b^2\}$	$\{a^2, a^2b, a^2b^2\}$	$\{ab^2, a^2, b\}$
$\{s, sa, sa^2\}$	$\{s, sb, sb^2\}$	$\{s, sab, sa^2b^2\}$
$\{sb, sab, sa^2b\}$	$\{sa, sab, sab^2\}$	$\{sa, sa^2b, sb^2\}$
$\{sb^2, sab^2, sa^2b^2\}$	$\{sa^2, sa^2b, sa^2b^2\}$	$\{sab^2, sa^2, sb\}$

Now the action of $\theta_3(x_3) = a^{\pm 1}b$ on the $\langle a \rangle$-cosets give the orbits

$$\{[1], [b], [b^2]\}, \quad \{[s], [sa], [sab^2]\}$$

and similarly on the $\langle b \rangle$-cosets we get the orbits

$$\{[1], [a], [a^2]\}, \quad \{[s], [sa^2], [sa]\}$$

neither of them giving any cone points. When acting on the $\langle ab \rangle$-cosets we get the orbits

$$\{[1]\}, \quad \{[a]\}, \quad \{[ab^2]\}, \quad \{[s], [sa], [sab^2]\}$$

giving only three cone points.

Again, the action of $\theta_3(x_4) = b^2$ on the $\langle a \rangle$- and $\langle ab \rangle$-cosets gives the orbits

$$\{[1], [b], [b^2]\}, \quad \{[s], [sa], [sab^2]\}$$

and

$$\{[1], [a], ab^2]\}, \quad \{[s], [sa], [sa^2]\}$$

respectively and thus not producing any cone points. The action of $\theta_3(x_4)$ on the $\langle b \rangle$-cosets, however, gives the orbits

$$\{[1]\}, \quad \{[a]\}, \quad \{[a^2]\}, \quad \{[s]\}, \quad \{[sa^2]\}, \quad \{[sa]\}$$

and hence producing six cone points of order 3. Then $s(\theta_3^{-1}(\langle b \rangle)) = (0; 3, 3, 3, 3, 3, 3)$. Hence, the surfaces $\mathbb{H}/Ker(\theta_3)$ are cyclic trigonal Riemann surfaces with unique central trigonal morphism.

$|G| = 21.$

[12] there are 4 subgroups of order 3 but two of them are conjugate

There are no Riemann surfaces of genus 4 with 21 automorphisms, since no Riemann surface of genus 4 admits an automorphism of order 7. Otherwise there would be a surface epimorphism $\hat{\theta} : \hat{\Delta} \to C_7$, with $s(\hat{\Delta}) = (1; 7)$, which is impossible.

$|G| = 24$

(i) As in the case (vii) of $|G| = 12$, groups with signature $(0; 2, 2, 2, 4)$ can not produce any trigonal Riemann surfaces since the orders of the elliptic generators of Δ_1 are all relative prime to 3.

(ii) For the signature $s(\Delta_2) = (0; 3, 4, 6)$: The only groups of order 24 generated by one element of order 3 and one element of order 6 are $A_4 \times C_2$ and $\langle 2, 3, 3\rangle = Q \rtimes C_3$[13], the binary tetrahedral group, see [16]. There are no possible epimorphisms from Δ_2 onto $A_4 \times C_2$ since this group has no elements of order 4. There are epimorphisms $\theta : \Delta_2 \to \langle 2, 3, 3\rangle$, for instance

$$\theta : \Delta_2 \to \langle 2, 3, 3\rangle \begin{cases} \theta(x_1) = sta \\ \theta(x_2) = s \\ \theta(x_3) = s^2a^2 \end{cases}$$

Now, the group $\langle 2, 3, 3\rangle$ contains just one conjugacy class of subgroups of order 3, and one conjugacy class of elements of order 6, with representatives sta and s^2a^2 respectively. Therefore it is sufficient to study the action of sta and s^2a^2 on the $(\langle a\rangle)$-cosets. The first action has orbits:

$$\{[a], [s], [t^3]\}, \quad \{[t], [s^2], [s^3]\}, \quad \{[ts]\}, \quad \{[st]\},$$

the second action has orbits:

$$\{[a], [s^2]\}, \quad \{[s], [t^3], [st], [s^3], [t], [ts]\}.$$

Thus the signature of $\theta^{-1}(\langle a\rangle)$ is $(1; 3, 3, 3)$ and the corresponding surface is not trigonal.

(iii) There are no epimorphisms from a Fuchsian group with signature $s(\Delta_3) = (0; 4, 4, 4)$ onto a group of order 24. This is impossible because there is no group of order 24 generated by two elements of order four whose product has order four. See [16].

(iv) There are no epimorphisms from a Fuchsian group with signature $s(\Delta_4) = (0; 3, 3, 12)$ onto a group of order 24 generated by two elements of order 3. This is impossible because the only such group is the binary tetrahedral group, which does not have elements of order 12. See [16].

[13] $\langle 2, 3, 3\rangle = \langle a, s, t | a^3 = t^4 = s^4 = (st)^4 = 1, s^2 = t^2, a^2sa = t, a^2ta = st\rangle$ and $Q = \langle s, t | S^4 = t^4 = 1, s^2 = t^2\rangle$, the quaternion group

(v) There are no epimorphisms from a Fuchsian group with signature $s(\Delta_5) = (0; 2, 8, 8)$ onto a group of order 24 since there is no group of order 24 generated by two elements of order 8. See [16].

(vi) Groups with signature $s(\Delta_6) = (0; 2, 6, 12)$ are non-maximal. Groups Δ_6 admit epimorphisms only onto $D_4 \times C_3$[14]. The epimorphisms $\overline{\theta} : \Delta_6 \to D_4 \times C_3$ with $\overline{\theta}(x_1) = s$ and $\overline{\theta}(x_2) = s\overline{a}$ lift to epimorphisms $\theta : \Delta \to C_3 \times \Sigma_4$, where $s(\Delta) = (0; 2, 3, 12)$, that are studied in the first case of $|G| = 72$.

$|G| = 27$.

There are no Riemann surfaces of genus 4 with 27 automorphisms. Otherwise it would be an epimorphism from a Fuchsian group with signature $s(\Delta) = (0; 3, 3, 9)$ onto $C_9 \rtimes C_3$. This group is not generated by order three elements. See [16].

$|G| = 30$.

As in the case (i) of $|G| = 15$, groups with signature $s(\Delta_1) = (0; 5, 5, 5)$ are non-maximal normal subgroups of groups with signature $(0; 2, 5, 10)$ and any epimorphisms would have to exist in the subgroups in order to exist.

$|G| = 36$

(i) First, consider Fuchsian groups Δ_1 with signature $(0; 2, 4, 12)$. The only groups of order 36 containing elements of order 12 are C_{36}, $C_{12} \times C_3$, $(C_3 \times C_3) \rtimes_1 C_4$ [15] and $C_3 \rtimes C_{12}$[16]. None of these groups are generated by elements of order 2 and 4. Hence there is no epimorphism from Δ_1 onto a group of order 36 and there are no cyclic trigonal Riemann surfaces of genus 4 such that the quotient $X_4/Aut(X_4)$ is uniformized by Δ_1.

(ii) Secondly, consider Fuchsian groups Δ_2 with signature $(0; 2, 2, 2, 3)$. The only group of order 36 generated by 3 involutions is $D_3 \times D_3$[17]. $D_3 \times D_3$ has three classes of subgroups[18] of order 3 represented by $\langle a \rangle$, $\langle b \rangle$ and $\langle ab \rangle$, where only the last one is non-normal.

The cosets are given by

[14] $D_4 \times C_3 = \langle \overline{a}, s | \overline{a}^{12} = s^2 = s\overline{a}s\overline{a}^5 = 1 \rangle$
[15] $(C_3 \times C_3) \rtimes_1 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = [a, b] = 1, t^3at = a^{-1}, t^3bt = b^{-1} \rangle$
[16] $C_3 \rtimes C_{12} = \langle a, b | a^3 = b^{12} = b^{-1}aba = 1 \rangle$
[17] $D_3 \times D_3 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 = [a, b] = [s, b] = [t, a] = (sa)^2 = (tb)^2 = 1 \rangle$
[18] Again, this group has 4 subgroups, but only 3 of them are non-conjugate.

$\langle\tau_1 = a\rangle$	$\langle\tau_2 = b\rangle$	$\langle\tau_3 = ab\rangle$
$\{1, a, a^2\}$	$\{1, b, b^2\}$	$\{1, ab, a^2b^2\}$
$\{b, ab, a^2b\}$	$\{a, ab, ab^2\}$	$\{a, a^2b, b^2\}$
$\{b^2, ab^2, a^2b^2\}$	$\{a^2, a^2b, a^2b^2\}$	$\{ab^2, a^2, b\}$
$\{s, sa, sa^2\}$	$\{s, sb, sb^2\}$	$\{s, sab, sa^2b^2\}$
$\{sb, sab, sa^2b\}$	$\{sa, sab, sab^2\}$	$\{sa, sa^2b, sb^2\}$
$\{sb^2, sab^2, sa^2b^2\}$	$\{sa^2, sa^2b, sa^2b^2\}$	$\{sab^2, sa^2, sb\}$
$\{t, ta, ta^2\}$	$\{t, tb, tb^2\}$	$\{t, tab, ta^2b^2\}$
$\{tb, tab, ta^2b\}$	$\{ta, tab, tab^2\}$	$\{ta, ta^2b, tb^2\}$
$\{tb^2, tab^2, ta^2b^2\}$	$\{ta^2, ta^2b, ta^2b^2\}$	$\{tab^2, ta^2, tb\}$
$\{st, sta, sta^2\}$	$\{st, stb, stb^2\}$	$\{st, stab, sta^2b^2\}$
$\{stb, stab, sta^2b\}$	$\{sta, stab, stab^2\}$	$\{sta, sta^2b, stb^2\}$
$\{stb^2, stab^2, sta^2b^2\}$	$\{sta^2, sta^2b, sta^2b^2\}$	$\{stab^2, sta^2, stb\}$

Now, consider the epimorphism $\theta : \Delta_2 \to D_3 \times D_3$ defined by

$$\theta : \Delta_2 \to D_3 \times D_3 \begin{cases} \theta(x_1) = s \\ \theta(x_2) = tb \\ \theta(x_3) = sta \\ \theta(x_4) = a^2b \end{cases}$$

The action of $\theta(x_4) = a^2b$ on the $\langle a\rangle$- and $\langle b\rangle$-cosets gives the orbits

$$\{[1], [b], [b^2]\}, \quad \{[s], [sb], [sb^2]\}, \quad \{[t], [tb^2], [tb]\}, \quad \{[st], [stb^2], [stb]\},$$

and

$$\{[1], [a^2], [a]\}, \quad \{[s], [sa], [sa^2]\}, \quad \{[t], [ta^2], [ta]\}, \quad \{[st], [sta], [sta^2]\},$$

respectively and hence none of them produce cone points.
The action of $\theta(x_4) = a^2b$ on the $(\langle ab\rangle)$-cosets has the following orbits:

$$\{[1], [a], [b]\}, \quad \{[s]\}, \quad \{[sb]\}, \quad \{[sa]\},$$

$$\{[t]\}, \quad \{[tb]\}, \quad \{[ta]\}, \quad \{[st], [stb], [sta]\}.$$

Then $s(\theta^{-1}(\langle ab\rangle))$ contains six periods equal to 3 and by the Riemann-Hurwitz formula $s(\theta^{-1}(\langle ab\rangle)) = (0; 3, 3, 3, 3, 3, 3)$. In the same way the action of $\theta(x_4) = a^2b$ on the $(\langle a^2b\rangle)$-cosets has the following orbits:

$$\{[1]\}, \quad \{[b]\}, \quad \{[ab]\}, \quad \{[s], [sa^2], [sb]\},$$

$$\{[t], [tb], [ta^2]\}, \quad \{[st]\}, \quad \{[stb]\}, \quad \{[sta^2]\}.$$

Again, $s(\theta^{-1}(\langle a^2b\rangle))$ contains six periods equal to 3 and again by the Riemann-Hurwitz formula $s(\theta^{-1}(\langle a^2b\rangle)) = (0; 3, 3, 3, 3, 3, 3)$. Thus the Riemann surfaces uniformized by $Ker(\theta)$ are cyclic trigonal Riemann surfaces that admit two different trigonal morphisms

$$f_1 : \mathbb{H}/Ker(\theta) \to \widehat{\mathbb{C}} \quad \text{and} \quad f_2 : \mathbb{H}/Ker(\theta) \to \widehat{\mathbb{C}}$$

induced by the subgroups $\langle ab\rangle$ and $\langle a^2b\rangle$ of $D_3 \times D_3$. By theorem (2.1.4) this is the unique conjugacy class of order 3 subgroups of $D_3 \times D_3$ inducing trigonal morphisms on the surfaces.

(iii) Fuchsian groups with signature $s(\Delta_3) = (0; 3, 3, 6)$ are subgroups of Fuchsian groups with signature $s(\Delta) = (0; 2, 3, 12)$ and so they are studied in the case $|G| = 72$ (i).

(iv) Fuchsian groups with signature $s(\Delta_4) = (0; 3, 4, 4)$ are subgroups of Fuchsian groups with signature $s(\Delta) = (0; 2, 4, 6)$ and they are studied in the case $|G| = 72$ (ii).

(v) Fuchsian groups with signature $s(\Delta_5) = (0; 2, 6, 6)$ are extensions of groups with signature $s(\Lambda) = (0; 3, 6, 6)$ (see case $|G| = 18$ (iii)).

Now, the non-maximal groups with signature $(0; 2, 6, 6)$ are (index 2) subgroups of groups Δ with signature $s(\Delta) = (0; 2, 4, 6)$. The epimorphism $\overline{\phi}_2 : \Delta_5 \to D_3 \times D_3$ is given by

$$\overline{\phi}_2 : \Delta_5 \to D_3 \times D_3 \begin{cases} \overline{\phi}_2(x_1) = stab^2 \\ \overline{\phi}_2(x_2) = sb \\ \overline{\phi}_2(x_3) = ta \end{cases}$$

The elevation of this epimorphism to $\theta : \Delta \to (C_3 \times C_3) \rtimes D_4$ is studied in case $|G| = 72$ (ii).

The epimorphism $\theta : \Delta_5 \to G$, where G is of order 36, is defined by an elevation of both $\overline{\theta}_1 : \Delta_3 \to C_6 \times C_3$ and $\overline{\theta}_2 : \Delta_3 \to C_3 \times D_3$ in case $|G| = 18$ (iii) to $\theta : \Delta_5 \to C_6 \times D_3$ [19]. The only possible elevation is $\theta : \Delta_5 \to C_6 \times D_3$ defined by

$$\theta : \Delta_5 \to C_6 \times D_3 \begin{cases} \theta(x_1) = s \\ \theta(x_2) = tab \\ \theta(x_3) = stab^2 \end{cases}$$

Note that $C_6 \times D_3$ has three classes of subgroups of order 3 represented by $\langle a\rangle$, $\langle b\rangle$ and $\langle ab\rangle$, where the last class contains two non-normal subgroups[20]. The coset tables in this case then becomes

[19] $C_6 \times D_3 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 = [s, t] = [a, b] = [s, b] = [t, a] = (sa)^2 = [t, b] = 1\rangle$, see [16].

[20] In fact, while $\langle a\rangle$ is normal the subgroup $\langle b\rangle$ is central in $C_6 \times D_3$.

$\langle\tau_1 = a\rangle$	$\langle\tau_2 = b\rangle$	$\langle\tau_4 = ab\rangle$
$\{1, a, a^2\}$	$\{1, b, b^2\}$	$\{1, ab, a^2b^2\}$
$\{b, ab, a^2b\}$	$\{a, ab, ab^2\}$	$\{a, a^2b, b^2\}$
$\{b^2, ab^2, a^2b^2\}$	$\{a^2, a^2b, a^2b^2\}$	$\{ab^2, a^2, b\}$
$\{s, sa, sa^2\}$	$\{s, sb, sb^2\}$	$\{s, sab, sa^2b^2\}$
$\{sb, sab, sa^2b\}$	$\{sa, sab, sab^2\}$	$\{sa, sa^2b, sb^2\}$
$\{sb^2, sab^2, sa^2b^2\}$	$\{sa^2, sa^2b, sa^2b^2\}$	$\{sab^2, sa^2, sb\}$
$\{t, ta, ta^2\}$	$\{t, tb, tb^2\}$	$\{t, tab, ta^2b^2\}$
$\{tb, tab, ta^2b\}$	$\{ta, tab, tab^2\}$	$\{ta, ta^2b, tb^2\}$
$\{tb^2, tab^2, ta^2b^2\}$	$\{ta^2, ta^2b, ta^2b^2\}$	$\{tab^2, ta^2, tb\}$
$\{st, sta, sta^2\}$	$\{st, stb, stb^2\}$	$\{st, stab, sta^2b^2\}$
$\{stb, stab, sta^2b\}$	$\{sta, stab, stab^2\}$	$\{sta, sta^2b, stb^2\}$
$\{stb^2, stab^2, sta^2b^2\}$	$\{sta^2, sta^2b, sta^2b^2\}$	$\{stab^2, sta^2, stb\}$

and using these we get the following actions.
The action of $\theta(x_2)$ on the $\langle a\rangle$-cosets gives the following orbits

$$\{[1], [tb], [b^2], [t], [b], [tb^2]\} \text{ and } \{[s], [stb], [sb^2], [st], [sb], [stb^2]\}.$$

The action of $\theta(x_2)$ on the $\langle b\rangle$-cosets gives the orbits

$$\{[1], [ta], [a^2], [t], [a], [ta^2]\} \text{ and } \{[s], [sta], [sa^2], [st], [sa], [sta^2]\}.$$

The action of $\theta(x_2)$ on the $\langle ab\rangle$-cosets gives the orbits

$$\{[1], [t]\},\ \{[a], [ta]\},\ \{[ab^2], [tab^2]\} \text{ and } \{[s], [sta], [sab^2], [st], [sa], [stb]\}.$$

Hence the action of $\theta(x_2)$ on either the $\langle a\rangle$- or $\langle b\rangle$-cosets does not leaves any fixed points. The action of $\theta(x_2)$ on the $\langle ab\rangle$-cosets leaves only three fixed points.
The action of $\theta(x_3)$ on the $\langle a\rangle$-cosets gives the orbits

$$\{[1], [stb], [b^2], [st], [b], [stb^2]\} \text{ and } \{[s], [tb], [sb^2], [t], [sb], [tb^2]\}.$$

leaving no fixed points.
The action of $\theta(x_3)$ on the $\langle b\rangle$-cosets gives the orbits

$$\{[1], [sta]\},\ \{[a], [sta^2]\},\ \{[a^2], [st]\},\ \{[s], [ta^2]\},\ \{[sa], [t]\} \text{ and } \{[sa^2], [ta]\}.$$

leaving six fixed points.
Finally, the action of $\theta(x_3)$ on the $\langle ab\rangle$-cosets gives the orbits

$$\{[1], [st], [a], [sta], [ab^2], [stab^2]\} \text{ and } \{[s], [ta], [sa], [tab^2], [sab^2], [t]\}.$$

leaving no fixed points.

Hence, the only subgroup producing six cone points is $\langle b \rangle$ and so $s(\theta^{-1}(\langle b \rangle)) = (0; 3, 3, 3, 3, 3, 3)$. Thus the surface[21] $\mathbb{H}/Ker(\theta)$ is a cyclic trigonal Riemann surface admitting a unique central trigonal morphism.

$|G| = 42$

As case $|G| = 21$, there are no Riemann surfaces of genus 4 with 42 automorphisms, since no Riemann surface of genus 4 admits an automorphism of order 7.

$|G| = 45$

As in case $|G| = 30$ (i), consider the signature $s(\Delta_1) = (0; 3, 3, 5)$. The groups with signature $s(\Delta_1) = (0; 5, 5, 5)$ are non-maximal normal subgroups of groups with signature $(0; 3, 3, 5)$ and any epimorphisms would have to exist in the subgroups in order to exist.

$|G| = 48$

(i) The groups with signature $s(\Delta_1) = (0; 2, 3, 24)$ contain normal subgroups with signature $(0; 3, 3, 12)$. In the case $|G| = 24$ (iv), we saw that this can not produce any epimorphism and so the extension of it will not produce any either.

(ii) The groups with signature $s(\Delta_2) = (0; 2, 4, 8)$ contain normal subgroups with signatures $(0; 4, 4, 4)$ (see case $|G| = 24$ (iii)) and $(0; 2, 8, 8)$ (see case $|G| = 24$ (v)). We have seen that these can not produce any epimorphism and so the extension of it will not produce any epimorphisms either.

$|G| = 54$

Again, the groups with signature $s(\Delta_1) = (0; 2, 3, 18)$ contain normal subgroups with signature $(0; 3, 3, 9)$. Since these do not produce any epimorphisms neither will their extensions.

$|G| = 60$

(i) Clearly there is no epimorphism from a Fuchsian group Δ_1 with signature $(0; 2, 3, 15)$ onto a group of order 60, since the only group of order 60 generated by elements of order 2 and 3 is A_5 and this does not contain elements of order 15.

(ii) The signature $s(\Delta_2) = (0; 2, 5, 5)$ cannot give trigonal surfaces since the orders of the elliptic generators of Δ are relative prime to 3.

$|G| = 72$

[21]The fact that this is a single surface will be proved in section 2.4.

(i) Consider Fuchsian groups Δ_1 with signature $(0;2,3,12)$. The group Δ_1 contains the group Λ_1 with signature $s(\Lambda_1) = (0;3,3,6)$ as a subgroup of index 2. Any epimorphism $\theta_1 : \Delta_1 \to G_{72}$ is an extension of an epimorphism $\phi_1 : \Lambda_1 \to G_{36}$. The existence of epimorphisms ϕ_1 obliges the group G_{36} to be generated by two elements of order 3. The only group of order 36 generated by elements of order 3 is $A_4 \times C_3$[22]. It contains three conjugacy classes of subgroups of order 3 with representatives $\langle a\rangle$, $\langle ab\rangle$ and $\langle b\rangle$, where $\langle b\rangle$ is central. The cosets are given by

$\langle \tau_1 = a\rangle$	$\langle \tau_2 = ab\rangle$	$\langle \tau_3 = b\rangle$
$\{1, a, a^2\}$	$\{1, ab, a^2b^2\}$	$\{1, b, b^2\}$
$\{s, sa, sa^2\}$	$\{s, sab, sa^2b^2\}$	$\{s, sb, sb^2\}$
$\{a^2sa, sas, a^2s\}$	$\{a^2sa, sasb, a^2sb^2\}$	$\{a^2sa, a^2sab, a^2sab^2\}$
$\{asa^2, as, asa\}$	$\{asa^2, asb, asab^2\}$	$\{asa^2, asa^2b, asa^2b^2\}$
$\{b, ab, a^2b\}$	$\{b, ab^2, a^2\}$	$\{a, ab, ab^2\}$
$\{sb, sab, sa^2b\}$	$\{sb, sab^2, sa^2\}$	$\{a^2, a^2b, a^2b^2\}$
$\{a^2sab, sasb, a^2sb\}$	$\{a^2sab, sasb^2, a^2s\}$	$\{as, asb, asb^2\}$
$\{asa^2b, asb, asab\}$	$\{asa^2b, asb^2, asa\}$	$\{sa^2, sa^2b, sa^2b^2\}$
$\{b^2, ab^2, a^2b^2\}$	$\{b^2, a, a^2b\}$	$\{a^2s, a^2sb, a^2sb^2\}$
$\{sb^2, sab^2, sa^2b^2\}$	$\{sb^2, sa, sa^2\}$	$\{sa, sab, sab^2\}$
$\{a^2sab^2, sasb^2, a^2sb^2\}$	$\{a^2sab^2, sas, a^2s\}$	$\{asa, asab, asab^2\}$
$\{asa^2b^2, asb^2, asab^2\}$	$\{asa^2b^2, as, asa\}$	$\{sas, sasb, sasb^2\}$

Consider the epimorphism ϕ_1 given by

$$\phi_1 : \Lambda_1 \to A_4 \times C_3 \begin{cases} \phi_1(z_1) = ba \\ \phi_1(z_2) = bsa \\ \phi_1(z_3) = bsas \end{cases}$$

As $\langle b\rangle$ is central, the action of any element of order 3 on the $\langle b\rangle$-cosets gives four orbits, each with three cosets. The action of any element of order 6 has six orbits, with 2 cosets each. Then $\phi_1(z_3) = bsas$ induces six periods of order 3 in $\phi_1^{-1}(\langle b\rangle)$. By the Riemann-Hurwitz formula $s(\phi_1^{-1}(\langle b\rangle)) = (0;3,3,3,3,3,3)$. Therefore, a cyclic trigonal Riemann surface X_4 will be uniformized by $Ker(\theta_1)$, where $\theta_1 : \Delta_1 \to G_{72}$ as above and G_{72} an extension of degree 2 of $A_4 \times C_3$ containing elements of order 12 . This extension is $\Sigma_4 \times C_3$[23], with $\langle b\rangle$ central in $\Sigma_4 \times C_3$. The epimorphism $\theta_1 : \Delta_1 \to \Sigma_4 \times C_3$ given by

$$\theta_1 : \Lambda_1 \to \Sigma_4 \times C_3 \begin{cases} \theta_1(x_1) = \overline{s} \\ \theta_1(x_2) = ab \\ \theta_1(x_3) = a^2\overline{s}b \end{cases}$$

[22] $A_4 \times C_3 = \langle a, b, s | a^3 = b^3 = s^2 = [a, b] = [s, b] = (as)^3 = 1\rangle$.
[23] $\Sigma_4 \times C_3 = \langle a, b, \overline{s} | a^3 = b^3 = (\overline{s})^2 = [a, b] = (a\overline{s})^4 = [\overline{s}, b] = 1\rangle$

yields a trigonal Riemann surface $X_4 = \mathbb{H}/Ker(\theta_1)$ with trigonal morphism induced by $\langle b\rangle$. But by theorem 1 in [18] this trigonal morphism is unique. Observe that the epimorphisms ϕ_1, θ_1 are unique.

(ii) Consider Fuchsian groups Δ_2 with signature $(0; 2, 4, 6)$. The group Δ_2 contains the groups Λ_2 with signature $s(\Lambda_2) = (0; 3, 4, 4)$ and some groups $\overline{\Lambda}_2$ with signature $s(\overline{\Lambda}_2) = (0; 2, 6, 6)$ as subgroup of index 2. Any epimorphism $\theta_2 : \Delta_2 \to G_{72}$ is an extension of epimorphisms $\phi_2 : \Lambda_2 \to G_{36}$, $\overline{\phi}_2 : \overline{\Lambda_2} \to D_3 \times D_3$. The existence of epimorphisms ϕ_2 obliges the group G_{36} to be generated by two elements of order 4. The only group of order 36 generated by elements of order 4 is $(C_3 \times C_3) \rtimes_2 C_4$[24]. It contains two conjugacy classes of subgroups[25] of order 3 with representatives $\langle a\rangle$ and $\langle ab\rangle$. The cosets are given by

$\langle \tau_1 = a\rangle$	$\langle \tau_2 = ab\rangle$
$\{1, a, a^2\}$	$\{1, ab, a^2b^2\}$
$\{b, ab, a^2b\}$	$\{b, ab^2, a^2\}$
$\{b^2, ab^2, a^2b^2\}$	$\{b^2, a, a^2b\}$
$\{t, b^2t, bt\}$	$\{t, ab^2t, a^2bt\}$
$\{at, ab^2t, abt\}$	$\{at, a^2b^2t, bt\}$
$\{a^2t, a^2b^2t, a^2bt\}$	$\{a^2t, b^2t, abt\}$
$\{t^2, a^2t^2, at^2\}$	$\{t^2, a^2b^2t^2, abt^2\}$
$\{bt^2, a^2bt^2, abt^2\}$	$\{bt^2, a^2t^2, ab^2t^2\}$
$\{b^2t^2, a^2b^2t^2, ab^2t^2\}$	$\{b^2t^2, a^2bt^2, at^2\}$
$\{t^3, bt^3, b^2t^3\}$	$\{t^3, a^2bt^3, ab^2t^3\}$
$\{at^3, abt^3, ab^2t^3\}$	$\{at^3, bt^3, a^2b^2t^3\}$
$\{a^2t^3, a^2bt^3, a^2b^2t^3\}$	$\{a^2t^3, abt^3, b^2t^3\}$

Consider the epimorphism ϕ_2

$$\phi_2 : \Lambda_2 \to (C_3 \times C_3) \rtimes C_4 \begin{cases} \phi_2(y_1) = a \\ \phi_2(y_2) = at \\ \phi_2(y_3) = bt^3 \end{cases}$$

The action of $\phi_2(y_1) = a$ on the $\langle a\rangle$-cosets has the following orbits:

$$\{[1]\}, \quad \{[b]\}, \quad \{[b^2]\}, \quad \{[t], [at], [a^2t]\},$$
$$\{[bt^2]\}, \quad \{[t^2]\}, \quad \{[b^2t^2]\} \quad \{[t^3], [at^3], [a^2t^3]\}.$$

The action of $\phi_2(y_1) = a$ on the $\langle b\rangle$-cosets has the same cycle structure.

The action of $\phi_2(y_1)$ on the $\langle ab\rangle$ gives the coset orbits

$$\{[1], [b^2], [b]\}, \quad \{[b]\}, \quad \{[b^2]\}, \quad \{[t], [at], [a^2t]\},$$

[24] $(C_3 \times C_3) \rtimes_2 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = [a, b] = 1, t^3at = b, t^3bt = a^{-1}\rangle$.

[25] This is because under conjugation with t, the elements of order 3 split up to two conjugacy classes. One contains ab, a^2b, a^2b^2, b^2a and the other class contains a, b, a^2, b^2.

$$\{[bt^2]\}, \quad \{[t^2]\}, \quad \{[b^2t^2]\} \quad \{[t^3],[at^3],[a^2t^3]\}.$$

Then $\phi_2^{-1}(\langle a\rangle)$ and $\phi_2^{-1}(\langle b\rangle)$ have six periods of order 3, induced by $\phi_2(y_1) = a$. By the Riemann-Hurwitz formula $s(\phi_2^{-1}(\langle a\rangle)) = s(\phi_2^{-1}(\langle b\rangle)) = (0;3,3,3,3,3,3)$. Therefore a cyclic trigonal Riemann surface Y_4 with non-unique trigonal morphism will be uniformized by $Ker(\theta_2)$, where $\theta_2 : \Delta_2 \to G_{72}$ as above and G_{72} is an extension of degree 2 of $(C_3 \times C_3) \rtimes C_4$ and of $D_3 \times D_3$ containing just two conjugacy classes of subgroups of order 3. This extension is $(C_3 \times C_3) \rtimes D_4$ [26]. The epimorphism $\theta_2 : \Delta_2 \to (C_3 \times C_3) \rtimes D_4$ given by

$$\theta_2 : \Delta_2 \to (C_3 \times C_3) \rtimes D_4 \begin{cases} \theta_2(x_1) = s \\ \theta_2(x_2) = ta \\ \theta_2(x_3) = stb \end{cases}$$

yields the required trigonal Riemann surface $Y_4 = \mathbb{H}/Ker(\theta_2)$ with non-unique trigonal morphisms. The trigonal morphisms

$$f_1 : \mathbb{H}/Ker(\theta_2) \to \widehat{\mathbb{C}} \quad \text{and} \quad f_2 : \mathbb{H}/Ker(\theta_2) \to \widehat{\mathbb{C}}$$

are induced by the conjugated subgroups $\langle a\rangle$ and $\langle b\rangle$ of $(C_3 \times C_3) \rtimes D_4$.

(iii) Fuchsian groups with signature $s(\Delta_3) = (0;3,3,4)$ contains normal subgroups with signature $(0;4,4,4)$. By case $|G| = 24$ (iii), there are no epimorphisms from groups with signature $(0;4,4,4)$ onto groups of order 24. Hence, there can be no epimorphisms from groups with signature $s(\Delta_3) = (0;3,3,4)$ onto groups of order 72, since these epimorphisms would have to be lifted from the ones from the lower case.

$|G| = 90$

Fuchsian groups with signature $s(\Delta_1) = (0;2,3,10)$ contains normal (index 6) subgroups with signature $(0;5,5,5)$. By case $|G| = 15$ (i), there are no epimorphisms from groups with signature $(0;5,5,5)$ onto groups of order 15. Hence, there can be no epimorphisms from groups with signature $s(\Delta_3) = (0;2,3,10)$ onto groups of order 90, since these epimorphisms would have to lift from the ones from the lower case.

$|G| = 108$

Similar to the case $|G| = 27$. Any Riemann surface of genus 4 can not have 108 automorphisms. This is because there is no Riemann surface of genus 4 having 27 automorphisms. Notice that the groups with signature $(0;3,3,9)$ are non-normal (index 4) subgroups of the Fuchsian groups with signature $(0;2,3,9)$.

[26] $(C_3 \times C_3) \rtimes D_4 = \langle a,b,t,s | a^3 = b^3 = t^4 = s^2 = (st)^2 = [a,b] = (sa)^2 = (sb)^2 = 1, t^3at = b, t^3bt = a^2\rangle$.

$|G| = 120$

The signature $s(\Delta) = (0; 2, 4, 5)$ cannot give trigonal surfaces since the orders of the elliptic generators of Δ are relative prime to 3.

$|G| = 144$

As a consequence of cases $|G| = 24$ (iii) and (v), there is no epimorphism from Δ, with signature $s(\Delta) = (0; 2, 3, 8)$, onto a group of order 144. Notice again, that the groups with signature $(0; 2, 8, 8)$ are non-normal (index 6) subgroups of groups with signature $(0; 2, 3, 8)$.

$|G| = 252$

As we have seen, Riemann surfaces with maximal automorphism groups are called Hurwitz surfaces, and in the case of genus 4, the Hurwitz bound is attained when the order of the automorphism group is 252. Thus, surfaces with automorphism group of order 252 would be a Hurwitz surfaces.

Here we will present a proof that there are no Hurwitz surfaces of genus 4.

First of all, there are several groups of order 252 (46 of them). Instead of investigating epimorphisms from the Fuchsian groups with signature $(0; 2, 3, 7)$ onto groups of order 252, we will prove that there exist no epimorphism from the non-normal Fuchsian subgroup Δ with signature $(0; 2, 7, 7)$ onto a group of order 28. Notice that the Fuchsian groups Δ has index 9 in the Fuchsian groups with signature $(0; 2, 3, 7)$.

There are only 4 groups of order 28, namely C_{28}, $C_{14} \times C_2$, D_{14} and $D_7 \times C_2$[27].

However, C_{28}, $C_{14} \times C_2$ and D_{14} are not generated by elements of order 7. Hence, in any of these cases there can not be any epimorphisms.

For the case $\Delta(0; 2, 7, 7) \to \langle 2, 2, 7\rangle$; Elements of order 7 are $s^{\pm 1}$, $s^{\pm 2}$, $s^{\pm 3}$ and so the order of $\theta(x_2x_3)$ is either 7 or 1. But for this to be an epimorphism the order of $\theta(x_2x_3)$ must equal the order of $\theta(x_1)$, that is 2. Hence there can be no epimorphism in this case.

Since there is no epimorphism from a group with signature $(0; 2, 7, 7)$ into a group of order 28 the signature $(0; 2, 3, 7)$ can not produce a surface of genus 4 with automorphism group of maximal order 252=84(4-1).

□

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$C_{28} = \langle u | u^{28} = 1\rangle$,
$C_{14} \times C_2 = \langle s, t | s^{14} = t^2 = [s, t] = 1\rangle$,
$D_{14} = \langle s, t | s^{14} = t^2 = (st)^2 = 1\rangle$,
$D_7 \times C_2 = s, t, u | s^7 = t^2 = u^2 = (st)^2 = [s, u] = [t, u] = 1\rangle$.

2.3 Cyclic trigonal Riemann surfaces of genus 4 with non-unique morphisms

The following results appear in

[13] A. Costa, M. Izquierdo, D. Ying, *On Riemann surfaces with non-unique cyclic trigonal morphism*, Manuscripta Math **118**, (2005) 443-453

We emphasize the results of theorem (2.2.2) by stating the following theorem about the cyclic trigonal Riemann surfaces admitting non-unique trigonal morphisms. This theorem show that Accola's bound is sharp for trigonal Riemann surfaces of genus 4.

Theorem 2.3.1. *[13] Using the same notation as in the proof of theorem (*2.2.2*)*

1. *There is a uniparametric family of Riemann surfaces $X_4(\lambda)$ of genus 4 admitting several cyclic trigonal morphisms.*
2. *The surfaces $X_4(\lambda)$ have $G = Aut(X_4(\lambda)) = D_3 \times D_3$ and the quotient Riemann surfaces $X_4(\lambda)/G$ are uniformized by the Fuchsian groups Δ with signature $s(\Delta) = (0; 2, 2, 2, 3)$.*
3. *There is one Riemann surface Y_4 in the family with automorphism group $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$ and the quotient Riemann surface $Y_4/Aut(Y_4)$ is uniformized by the Fuchsian group $\overline{\Delta}$ with signature $s(\overline{\Delta}) = (0; 2, 4, 6)$.*

Proof. By case $|G| = 36$ (ii) in the proof of theorem (2.2.2), The Riemann surfaces uniformized by $Ker(\theta)$ are cyclic trigonal Riemann surfaces that admit two different trigonal morphisms f_1 and f_2 induced by the subgroups $\langle ab \rangle$ and $\langle a^2 b \rangle$ of $D_3 \times D_3$. The dimension of the family of surfaces $\mathbb{H}/Ker(\theta)$ is given by the dimension of the space of groups Δ_2 with $s(\Delta_2) = (0; 2, 2, 2, 3)$. This has complex dimension $3 \cdot 0 - 3 + 4 = 1$. This proves the first and second statement of the theorem.
For the third statement, the epimorphism $\theta_2 : \Delta_2 \to (C_3 \times C_3) \rtimes D_4$ given in case $|G| = 72$ (ii) yields the required trigonal Riemann surface $Y_4 = \mathbb{H}/Ker(\theta_2)$ with non-unique trigonal morphisms. Since $(C_3 \times C_3) \rtimes D_4$ is an extension of degree 2 of $D_3 \times D_3$ this obliges the surface Y_4 to lie inside the family $X_4(\lambda)$. $\square$

2.4 The equisymmetric strata of trigonal Riemann surfaces of genus 4

The following results appear in

[14]A. Costa, M. Izquierdo, D. Ying, *On the family of cyclic trigonal Riemann surfaces of genus 4 with several trigonal morphisms*, Submitted 2006

[22] M. Izquierdo, D. Ying, *On the space of Cyclic trigonal Riemann surfaces of genus 4*, To appear in Groups St. Andrews 2005, London Math. Soc. Lecture notes, Cambridge University Press, (2006).

[23] M. Izquierdo, D. Ying, *Equisymmetric strata of the moduli space of cyclic trigonal Riemann surfaces of genus 4*, Submitted 2006

Using the action of the finite group $G = Aut(X)$ on X itself we can obtain information of the singularities of the moduli space of trigonal Riemann surfaces and also information about the moduli group. First we will recall some of the terminology.
We have seen in section 1.5 that two surfaces are called **equisymmetric** if the two surfaces' conformal automorphism groups determine conjugate finite subgroups of the mapping class group. Also, the subset of the moduli space corresponding to surfaces equisymmetric with given surface forms a locally closed subvariety of the moduli space, called an **equisymmetric stratum**.
Recall from lemma (1.5.3) that two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ are topologically equivalent if there exist two automorphisms $\phi : \Delta \to \Delta$ and $\omega : G \to G$ such that $\theta_2 = \omega \circ \theta_1 \circ \phi^{-1}$. That is, making the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\theta_1} & G \\ \phi \downarrow & & \downarrow \omega \\ \Delta & \xrightarrow{\theta_2} & G \end{array}$$

commute.
With other words, let $\mathcal{B}$ be the subgroup of $Aut(\Delta)$ induced by orientation preserving homeomorphisms. Then two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define the same class of G-actions if and only if they lie in the same $\mathcal{B} \times Aut(G)$-class. See [8].
To get an algebraic characterization of $\mathcal{B}$, consider the group

$$\overline{\Delta} = \langle \overline{x}_1, \ldots, \overline{x}_r | \overline{x}_1 \ldots \overline{x}_r = 1 \rangle \qquad (2.7)$$

$\overline{\Delta}$ is the fundamental group of the punctured surface X_0 obtained by removing the r branch points of the quotient Riemann sphere X/G. $\mathcal{B}$ can be identified with a certain subgroup of the mapping class group of X_0.

Now, any automorphism $\phi \in \Delta$ can be lifted to an automorphism $\overline{\phi} \in \overline{\Delta}$ such that for $1 \leq j \leq r$, $\overline{\phi}(\overline{x}_j)$ is conjugate to some $(\overline{x}_j)$. The induced representation $\mathcal{B} \to \Sigma_r$ preserves the branching orders.
We are interested in finding elements of $\mathcal{B} \times Aut(G)$ that make our epimorphisms $\theta_1, \theta_2 : \Delta \to G$ equivalent. We can produce the automorphism $\phi \in \mathcal{B}$ ad hoc. In our case the only elements $\mathcal{B}$ we need are compositions of $x_j \to x_{j+1}$ and $x_{j+1} \to x_{j+1}^{-1} x_j x_{j+1}$, where we write down only the action on the generators moved by the automorphism. See [8]. Of course the induced representation above, induced by these movements, must preserve the branching orders.

Equisymmetric Riemann surfaces and actions of finite groups

Before starting with the examination of the moduli spaces the following remark should be made clear. It is a direct conclusion of the proposition 1.3.37

Remark 2.4.1. *If $X = \mathbb{H}/\Gamma$, with Γ a Fuchsian surface group with signature $s(\Gamma) = (0; m_1, \ldots, m_r)$ then the existence of the epimorphisms $\theta : \Delta \to G = Aut(X)$ imposes that*

$$Order(\theta(x_i)) = m_i \quad i = 1, \ldots r,$$

and the product $\theta(x_1)\theta(x_2) \cdots \theta(x_{r-1}) = \theta(x_r)$.

Proposition 2.4.2. *[18] The space of $\mathcal{M}_4^3$ of cyclic trigonal Riemann surfaces of genus 4 form a disconnected subspace of dimension 3 of the moduli space $\mathcal{M}_4$.*

Proof. By case $|G| = 3$ in the proof of theorem (2.2.2) there exists epimorphisms $\theta : \Delta \to C_3$ where $s(\Delta) = (0; 3, 3, 3, 3, 3, 3)$. Thus we can either have an epimorphism of the type

$$\theta_1 : \Delta \to C_3 \left\{ \begin{array}{l} \theta_1(x_{l_i}) = a \\ \theta_1(x_{k_i}) = a^{-1} \\ i = 1, \ldots 3 \end{array} \right.$$

where half of the stabilizers $(\theta_1(x_{k_i}))$ of the cone point rotate in opposite direction, or we can have

$$\theta_2 : \Delta \to C_3 \left\{ \begin{array}{l} \theta_2(x_i) = a \\ i = 1, \ldots, 6 \end{array} \right.$$

where all the stabilizers of the cone points rotate in the same direction.
Now, the space of trigonal Riemann surfaces of genus 4 has (complex) dimension given by the dimension of the space of groups with signature $(0; 3, 3, 3, 3, 3, 3)$ and this is $d(\Delta) = 3 \cdot 0 - 3 + 6 = 3$.

By theorem 2.(i) in ([18]) the space $\mathcal{M}_4^3$ is disconnected. It consists of two components $\mathcal{C}_1$ and $\mathcal{C}_2$, each given by the epimorphisms θ_1 and θ_2 respectively. □

Proposition 2.4.3. *The subspace* ${}^6\mathcal{M}_4^3$*, determined by the Fuchsian groups* Δ'' *with* $s(\Delta'') = (0; 2, 2, 3, 3, 3)$*, formed by Riemann surfaces of genus 4 with automorphism group of order 6 is a disconnected space of dimension 2 consisting of two connected components* $\mathcal{C}_1^6$*,* $\mathcal{C}_2^6$ *consisting of trigonal Riemann surfaces with automorphism groups* D_3 *and* C_6 *respectively.* $\mathcal{C}_i^6 \subset \mathcal{C}_i$*.*

Proof. From the proof of theorem (2.2.2) where $|G| = 6$ there are possible epimorphisms from Fuchsian groups Δ_2 with signature $s(\Delta_2) = (0; 2, 2, 3, 3, 3)$ onto both C_6 and D_3.
The epimorphisms $\theta : \Delta_2 \to C_6$ in this case are given by

$$\theta : \Delta_2 \to C_6 \begin{cases} \theta(x_1) = a^3 \\ \theta(x_2) = a^3 \\ \theta(x_3) = a^2 \\ \theta(x_4) = a^2 \end{cases} \text{or} \quad \begin{cases} \theta(x_1) = a^3 \\ \theta(x_2) = a^3 \\ \theta(x_3) = a^{-2} \\ \theta(x_4) = a^{-2} \end{cases}$$

but, up to renaming the generator in C_6, this is unique. Note that the mapping of $\theta(x_5)$ is uniquely determined by the images of $x_1, \ldots, x_4$ and so is omitted.
For the other epimorphism, $\theta : \Delta \to D_3 = \langle a, s | a^3 = s^2 = (sa)^2 = 1 \rangle$ the situation is more complicated. Generally the epimorphism is given by

$$\theta : \Delta_2 \to C_6 \begin{cases} \theta(x_1) = sa^i \\ \theta(x_2) = sa^j \\ \theta(x_3) = a^k \\ \theta(x_4) = a^l \end{cases}$$

where $i, j \in \{0, 1, 2\}$ and $k, l \in \{1, 2\}$ such that $j - i + k + l \not\equiv 0 \mod (3)$
By conjugations in D_3 each of these epimorphisms are conjugated to one of the following three types:

$$\theta_1 \begin{cases} \theta_1(x_1) = s \\ \theta_1(x_2) = s \\ \theta_1(x_3) = a \\ \theta_1(x_4) = a \end{cases}, \quad \theta_2 \begin{cases} \theta_2(x_1) = s \\ \theta_2(x_2) = sa \\ \theta_2(x_3) = a \\ \theta_2(x_4) = a^2 \end{cases}, \quad \theta_3 \begin{cases} \theta_3(x_1) = s \\ \theta_3(x_2) = sa \\ \theta_3(x_3) = a^2 \\ \theta_3(x_4) = a^2 \end{cases}$$

Recall, from lemma (1.5.3), that two epimorphisms are topologically equivalent if there exists automorphisms $\phi : \Delta \to \Delta$ and $\omega : D_3 \to D_3$ such that

$$\theta_2 = \omega \circ \theta_1 \circ \phi^{-1}$$

This action is given as an element in the group $\mathcal{B} \times \mathrm{Aut}\,(D_3)$, and we will now produce such element ad hoc, showing that the three epimorphisms are topologically equivalent.

First, consider the element $\phi_{2,3} : \Delta \to \Delta$ given by

$$\phi_{2,3}\begin{cases} \phi(x_1) = x_1 \\ \phi(x_2) = x_3 \\ \phi(x_3) = x_3^{-1}x_2x_3 \\ \phi(x_4) = x_4 \\ \phi(x_5) = x_5 \end{cases}, \quad (\phi_{2,3})^2\begin{cases} \phi(x_1) = x_1 \\ \phi(x_2) = x_3^{-1}x_2x_3 \\ \phi(x_3) = x_3^{-1}x_2^{-1}x_3x_2x_3 \\ \phi(x_4) = x_4 \\ \phi(x_5) = x_5 \end{cases}$$

The composition of $(\phi_{2,3})^2$ with θ_1 becomes

$$\theta_1 \circ (\phi_{2,3})^2\begin{cases} \theta_1((\phi_{2,3})^2(x_1)) = s \\ \theta_1((\phi_{2,3})^2(x_2)) = a^{-1}sa = sa^2 \\ \theta_1((\phi_{2,3})^2(x_3)) = a^{-1}sasa = a^2 \\ \theta_1((\phi_{2,3})^2(x_4)) = a \end{cases}$$

Similarly, we look at the composition of θ_2 with an element $\omega : \mathrm{Aut}\,(D_3) \to \mathrm{Aut}\,(D_3)$. Consider the element ω_s given by

$$\omega_s\big[x \mapsto s^{-1}xs\big] : \begin{cases} \omega_s(s) = s \\ \omega_s(a) = a^2 \end{cases}$$

The composition $\omega_s \circ \theta_2$ becomes

$$\omega_s \circ \theta_2\begin{cases} \omega_s(\theta_2(x_1)) = s \\ \omega_s(\theta_2(x_2)) = sa^2 \\ \omega_s(\theta_2(x_3)) = a^2 \\ \omega_s(\theta_2(x_4)) = a \end{cases}$$

and thus, $\omega_s \circ \theta_2 = \theta_1 \circ (\phi_{2,3})^2$. Hence, the element $((\phi_{2,3})^2, \omega_s) \in \mathcal{B} \times \mathrm{Aut}\,(D_3)$ defines an equivalence between θ_1 and θ_2.
Similarly, consider the element $\theta_{3,4} : \Delta \to \Delta$ given by

$$\phi_{3,4}\begin{cases} \phi_{3,4}(x_1) = x_1 \\ \phi_{3,4}(x_2) = x_2 \\ \phi_{3,4}(x_3) = x_4 \\ \phi_{3,4}(x_4) = x_4^{-1}x_3x_4 \\ \phi_{3,4}(x_5) = x_5 \end{cases}$$

Composing $(\phi_{2,3})^2$ with $(\phi_{2,3})^2$ and then θ_1 yields

$$g = \phi_{3,4} \circ (\phi_{2,3})^2\begin{cases} g(x_1) = x_1 \\ g(x_2) = x_3^{-1}x_2x_3 \\ g(x_3) = x_4 \\ g(x_4) = x_4^{-1}x_3^{-1}x_2^{-1}x_3x_2x_3x_4 \\ g(x_5) = x_5 \end{cases}$$

$$\theta_1 \circ g\begin{cases} \theta_1(g(x_1)) = s \\ \theta_1(g(x_2)) = a^{-1}sa = sa^2 \\ \theta_1(g(x_3)) = a \\ \theta_1(g(x_4)) = a^{-1}a^{-1}sasaa = a^2 \end{cases}$$

Since this is still not the right expression we are looking for we need to apply $\phi_{2,3}$ twice, once more, and then we obtain

$$g' = (\phi_{2,3})^2 \circ \phi_{3,4} \circ (\phi_{2,3})^2 \begin{cases} g'(x_1)) = x_1 \\ g'(x_2)) = x_4^{-1}x_3^{-1}x_2x_3x_4 \\ g'(x_3)) = x_4^{-1}x_3^{-1}x_2^{-1}x_3x_4x_3^{-1}x_2x_3x_4 \\ g'(x_4)) = x_4^{-1}x_3^{-1}x_2^{-1}x_3x_2x_3x_4 \\ g'(x_5)) = x_5 \end{cases}$$

$$\theta_1 \circ g' \begin{cases} \theta_1(g'(x_1)) = s \\ \theta_1(g'(x_2)) = a^{-1}a^{-1}saa = sa \\ \theta_1(g'(x_3)) = a^{-1}a^{-1}saaa^{-1}saa = a^2 \\ \theta_1(g'(x_4)) = a^{-1}a^{-1}sasaa = a^2 \end{cases}$$

Hence we have that $\theta_3 = \theta_1 \circ (\phi_{2,3})^2 \circ \phi_{3,4} \circ (\phi_{2,3})^2$, and the element $((\phi_{2,3})^2 \circ \phi_{3,4} \circ (\phi_{2,3})^2, 1_d) \in \mathcal{B} \times \mathrm{Aut}\,(D_3)$ defines an equivalence between θ_1 and θ_3.
This shows that the epimorphisms are all topologically equivalent and thus defines topologically equivalent actions of G on the space of trigonal Riemann surfaces with automorphism group D_3.
Summing up we have, there are two strata, with complex dimension $d(\Delta) = 3g - 3 + r = 3 \cdot 0 - 3 + 5 = 2$ of cyclic trigonal Riemann surfaces of genus 4 with six automorphisms:

- one stratum is determined by the unique class of actions of D_3.
- one stratum is determined by the unique action of C_6.

The surfaces with automorphism group D_3 lie in $\mathcal{C}_1$ and the surfaces with automorphism group C_6 lie in $\mathcal{C}_2$.

□

Proposition 2.4.4. *The subspace ${}^{12}\mathcal{M}_4^3$, determined by the Fuchsian groups Δ' with $s(\Delta') = (0; 2, 2, 3, 6)$, formed by Riemann surfaces of genus 4 with automorphism group of order 12 is a disconnected space of dimension 1 consisting of two connected components $\mathcal{C}_1^{12}$, $\mathcal{C}_2^{12}$ consisting of trigonal Riemann surfaces with automorphism groups D_6 and C_{12} respectively. $\mathcal{C}_i^{12} \subset \mathcal{C}_i$.*

Proof. From the proof of theorem (2.2.2) where $|G| = 12$ there are possible epimorphisms from Fuchsian groups Δ_3 with signature $s(\Delta_3) = (0; 2, 2, 3, 6)$ onto both $C_6 \times C_2$ and D_6.
The epimorphisms $\theta : \Delta_3 \to C_6 \times C_2$ are given by

$$\theta : \Delta_3 \to C_6 \times C_2 \begin{cases} \theta(x_1) = s \\ \theta(x_2) = sa^3 \\ \theta(x_3) = a^2 \end{cases}, \quad \begin{cases} \theta(x_1) = s \\ \theta(x_2) = sa^3 \\ \theta(x_3) = a^{-2} \end{cases}$$

Up to renaming the generator (a) these two define the same epimorphism.

The epimorphisms $\theta : \Delta_3 \to D_6$ are given by

$$\theta_i : \Delta_3 \to D_6 \begin{cases} \theta(x_1) = sa^i \\ \theta(x_2) = sa^{i\pm 3} \\ \theta(x_3) = a^{\pm 2} \end{cases}$$

for $i = 1, \ldots, 5$. Up to counting the different signs in the exponents there are a total of twenty such epimorphisms. However, as we shall see they all belong to one class.

By defining elements ω_j and $\overline{\omega}_j$ as

$$\omega_j \left[x \mapsto (sa^j)^{-1} x sa^j \right] : \begin{cases} \omega_s(s) = sa^{2j} \\ \omega_s(a) = a^{-1} \end{cases}$$

and

$$\overline{\omega}_j \left[x \mapsto a^{-j} x a^j \right] : \begin{cases} \overline{\omega}_s(s) = sa^{2j} \\ \overline{\omega}_s(a) = a \end{cases}$$

we get possible ways of taking us from one epimorphism to another. The below scheme shows some of the possible actions that can arise.

$$\omega_0 \circ \begin{cases} \theta(x_1) = sa \\ \theta(x_2) = sa^4 \\ \theta(x_3) = a^2 \end{cases} = \begin{cases} \theta(x_1) = sa^5 \\ \theta(x_2) = sa^2 \\ \theta(x_3) = a^4 \end{cases}$$

$$\omega_1 \circ \begin{cases} \theta(x_1) = sa \\ \theta(x_2) = sa^4 \\ \theta(x_3) = a^2 \end{cases} = \begin{cases} \theta(x_1) = sa \\ \theta(x_2) = sa^4 \\ \theta(x_3) = a^4 \end{cases}$$

$$\omega_2 \circ \begin{cases} \theta(x_1) = sa \\ \theta(x_2) = sa^4 \\ \theta(x_3) = a^2 \end{cases} = \begin{cases} \theta(x_1) = sa^3 \\ \theta(x_2) = s \\ \theta(x_3) = a^4 \end{cases}$$

$$\overline{\omega}_1 \circ \begin{cases} \theta(x_1) = sa \\ \theta(x_2) = sa^4 \\ \theta(x_3) = a^2 \end{cases} = \begin{cases} \theta(x_1) = sa^3 \\ \theta(x_2) = s \\ \theta(x_3) = a^2 \end{cases}$$

$$\overline{\omega}_2 \circ \begin{cases} \theta(x_1) = sa \\ \theta(x_2) = sa^4 \\ \theta(x_3) = a^2 \end{cases} = \begin{cases} \theta(x_1) = sa^5 \\ \theta(x_2) = sa^2 \\ \theta(x_3) = a^2 \end{cases}$$

The above elements, $\omega_j, \overline{\omega}_j \in \mathrm{Aut}\,(D_6)$ can not alone produce all the epimorphisms. Instead there has to be a braid element permuting the two generators of order 2. This element $\phi_{1,2} \in \mathcal{B}$ is given by

$$\phi_{1,2} \begin{cases} \phi(x_1) = x_2 \\ \phi(x_2) = x_2^{-1} x_1 x^2 \\ \phi(x_3) = x_3 \\ \phi(x_4) = x_4 \end{cases}$$

Using the above elements we can show that the above epimorphisms all fall into one topological equivalence class.
Therefore, there are two strata, with complex dimension $d(\Delta_2) = 3g-3+r = 0-3+4 = 1$, of cyclic trigonal Riemann surfaces of genus 4 with 12 automorphisms:

- one stratum is determined by the unique class of actions of D_6,
- one stratum is determined by the unique action of $C_6 \times C_2$.

The surfaces with automorphism group D_6 lie in $\mathcal{C}_1$ and the surfaces with automorphism group $C_6 \times C_2$ lie in $\mathcal{C}_2$. □

Proposition 2.4.5. *There is one cyclic trigonal Riemann surface T_4 determined by the Fuchsian group Δ_4 with $s(\Delta_4) = (0; 3, 5, 15)$ and automorphism group C_{15}. $T_4 \in \mathcal{C}_2$*

Proof. From the proof of theorem (2.2.2) where $|G| = 15$ there are possible epimorphisms from Fuchsian groups Δ_2 with signature $s(\Delta_2) = (0; 3, 5, 15)$ onto C_{15}. The class of epimorphisms is given by

$$\theta : \begin{cases} \theta(x_1) = a^{\pm 5} \\ \theta(x_2) = a^{\pm 3i} \end{cases}$$

where $i = 1, 2$, and where the image of the last generator $\theta(x_3)$ is uniquely determined by the image of the first two. By counting the signs of the exponents we get 8 different epimorphisms:

$$\theta_1 : \begin{cases} \theta(x_1) = a^5 \\ \theta(x_2) = a^3 \end{cases} \quad \theta_2 : \begin{cases} \theta(x_1) = a^5 \\ \theta(x_2) = a^6 \end{cases} \quad \theta_3 : \begin{cases} \theta(x_1) = a^5 \\ \theta(x_2) = a^{-3} \end{cases}$$

$$\theta_4 : \begin{cases} \theta(x_1) = a^5 \\ \theta(x_2) = a^{-6} \end{cases} \quad \theta_5 : \begin{cases} \theta(x_1) = a^{-5} \\ \theta(x_2) = a^3 \end{cases} \quad \theta_6 : \begin{cases} \theta(x_1) = a^{-5} \\ \theta(x_2) = a^6 \end{cases}$$

$$\theta_7 : \begin{cases} \theta(x_1) = a^{-5} \\ \theta(x_2) = a^{-3} \end{cases} \quad \theta_8 : \begin{cases} \theta(x_1) = a^{-5} \\ \theta(x_2) = a^{-6} \end{cases}$$

The element $\omega_2 \in \mathrm{Aut}\,(C_{15})$ given by

$$\omega_2(a) = a^2$$

permutes the epimorphisms in the following way:

$$\omega_2 \circ \theta_1 = \theta_6, \quad \omega_2 \circ \theta_6 = \theta_3, \quad \omega_2 \circ \theta_3 = \theta_8, \quad \omega_2 \circ \theta_8 = \theta_1$$

The four other epimorphisms are given by simply renaming the generators, that is, taking a to a^{-1}.
Since there is only one unique action $(1_d, \omega_2) \in \mathcal{B} \times \mathrm{Aut}\,(C_{15})$, taking the epimorphisms into one equivalence class, and the dimension of the strata in this case is $d(\Delta_1) = 0 - 3 + 3 = 0$. There is a unique cyclic trigonal Riemann surface T_4 of genus 4 with $\mathrm{Aut}\,(T_4) = C_{15}$. □

Proposition 2.4.6. *The subspace* ${}^{18}\mathcal{M}_4^3$*, determined by the Fuchsian groups* Δ''' *with* $s(\Delta''') = (0; 2, 2, 3, 3)$*, formed by Riemann surfaces of genus 4 with automorphism group of order 18 is a connected space of dimension 1 consisting of trigonal Riemann surfaces with automorphism group* $D_3 \times C_3$*.* ${}^{18}\mathcal{M}_4^3 \subset \mathcal{C}_2$

Proof. Again, from the proof of theorem (2.2.2), where $|G| = 18$ there are possible epimorphisms from Fuchsian groups Δ_4 with signature $s(\Delta_4) = (0; 2, 2, 3, 3)$ onto $C_3 \times D_3$.
The possible epimorphisms are given by

$$\theta_{i,j} : \Delta_4 \to C_3 \times D_3 \begin{cases} \theta_{i,j}(x_1) = sa^i \\ \theta_{i,j}(x_2) = sa^j \\ \theta_{i,j}(x_3) = a^{i-j}b \end{cases} \quad \text{where } i \neq j \in \{0, 1, 2\}$$

where again image of the last generator has been omitted since it is uniquely determined by the images of the three first generators.
Using the conjugations $\omega_{a^h}, \omega_{sa^h} \in \mathrm{Aut}\,(C_3 \times D_3)$ given by

$$\omega_{a^h}(x) = a^{-h}xa^h$$

and

$$\omega_{sa^h}(x) = sa^h xsa^h$$

we can obtain every epimorphism. Listing the possible epimorphisms we get

$$\theta_1 \begin{cases} \theta_1(x_1) = s \\ \theta_1(x_2) = sa \\ \theta_1(x_3) = a^2b \end{cases}, \theta_2 \begin{cases} \theta_2(x_1) = s \\ \theta_2(x_2) = sa^2 \\ \theta_2(x_3) = ab \end{cases}, \theta_3 \begin{cases} \theta_3(x_1) = sa \\ \theta_3(x_2) = s \\ \theta_3(x_3) = ab \end{cases},$$

$$\theta_4 \begin{cases} \theta_4(x_1) = sa \\ \theta_4(x_2) = sa^2 \\ \theta_4(x_3) = a^2b \end{cases}, \theta_5 \begin{cases} \theta_5(x_1) = sa^2 \\ \theta_5(x_2) = s \\ \theta_5(x_3) = a^2b \end{cases}, \theta_6 \begin{cases} \theta_6(x_1) = sa^2 \\ \theta_6(x_2) = sa \\ \theta_6(x_3) = ab \end{cases}$$

Then

$$\omega_a \circ \theta_1 = \theta_5, \quad \omega_{a^2} \circ \theta_1 = \theta_4, \quad \omega_s \circ \theta_1 = \theta_2$$

$$\omega_a \circ \theta_2 = \theta_6, \quad \omega_{a^2} \circ \theta_2 = \theta_3,$$

and so the actions of $C_3 \times D_3$ are topologically equivalent and the cyclic trigonal Riemann surfaces determined by Δ_4 are equisymmetric. Hence there is one stratum of complex dimension $d(\Delta_4) = 0 - 3 + 4 = 1$ of cyclic trigonal Riemann surfaces of genus 4 with 18 automorphisms. The surfaces have automorphism group $C_3 \times D_3$. This family belongs to the component $\mathcal{C}_2$.
Recall from the results earlier that this uniparametric family of cyclic trigonal Riemann surfaces also have a unique central trigonal morphism. □

Proposition 2.4.7. *The connected subspace* ${}^{36}\mathcal{M}_4^3$ *of* $\mathcal{M}_4^3$ *formed by Riemann surfaces* $X_4(\lambda)$ *of genus 4 with automorphism group of order 36 is a space of dimension 1 determined by the Fuchsian groups* Δ *with* $s(\Delta) = (0; 2, 2, 2, 3)$. *The automorphism group of the Riemann surfaces is* $D_3 \times D_3$ *and the surfaces admit two trigonal morphisms.* ${}^{36}\mathcal{M}_4^3 \subset \mathcal{C}_1$

Proof. The proof of theorem (2.2.2), where $|G| = 36$, gives existence of an epimorphisms from the Fuchsian groups Δ_2 with signature $s(\Delta_2) = (0; 2, 2, 2, 3)$ onto $D_3 \times D_3$.
The general setting for such epimorphism is

$$\theta : \Delta_2 \to D_3 \times D_3 \begin{cases} \theta(x_1) = sa^i \\ \theta(x_2) = tb^j \\ \theta(x_3) = sta^h b^k \\ \theta(x_4) = a^\varepsilon b^\delta \end{cases}$$

The only way for θ to be an epimorphism is if $\theta(x_4) = a^{\pm1} b^{\pm1}$. To see this, note that the only elements of order 3 in $D_3 \times D_3$ are $a^{\pm1}b^{\pm1}$, $a^{\pm1}$ and $b^{\pm1}$. Hence if $\theta(x_4) = a^{\pm1}$ or $\theta(x_4) = b^{\pm1}$ then the action of $\theta(x_4)$ on the $\langle a \rangle$- or $\langle b \rangle$-cosets leaves twelve fixed cosets which is geometrically impossible.[28]
With this information we can now list the possible epimorphisms and divide them into 6 cases depending on the conjugacy classes of involutions of $D_3 \times D_3$.

$$\theta_1 : \begin{cases} \theta_1(x_1) = sa^i \\ \theta_1(x_2) = tb^j \\ \theta_1(x_3) = sta^h b^k \end{cases}, \quad \theta_2 : \begin{cases} \theta_2(x_2) = tb^j \\ \theta_2(x_1) = sa^i \\ \theta_2(x_3) = sta^h b^k \end{cases},$$

$$\theta_3 : \begin{cases} \theta_3(x_2) = tb^j \\ \theta_3(x_3) = sta^i b^k \\ \theta_3(x_1) = sa^h \end{cases}, \quad \theta_4 : \begin{cases} \theta_4(x_1) = sa^i \\ \theta_4(x_3) = sta^h b^j \\ \theta_4(x_2) = tb^k \end{cases},$$

$$\theta_5 : \begin{cases} \theta_5(x_3) = sta^i b^j \\ \theta_5(x_1) = sa^h \\ \theta_5(x_2) = tb^k \end{cases}, \quad \theta_6 : \begin{cases} \theta_6(x_3) = sta^i b^j \\ \theta_6(x_2) = tb^h \\ \theta_6(x_1) = sa^k \end{cases},$$

where $0 \le i, j \le 2$ such that $i \not\equiv h \mod (3)$ and $j \not\equiv k \mod (3)$.

1) *Within each case all the epimorphisms define the same action of* $D_3 \times D_3$ *on* $\mathbb{H}/Ker(\theta_i)$.

   To see this, note that the epimorphism of type θ_1 given by

$$\theta_0 : \begin{cases} \theta_0(x_1) = s \\ \theta_0(x_2) = t \\ \theta_0(x_3) = stab \end{cases}$$

[28]By the Riemann-Hurwitz formula, any Riemann surface with more than six cone points would have a negative genus which clearly is impossible.

is conjugate to the epimorphism

$$\theta : \begin{cases} \theta(x_1) = sa^i \\ \theta(x_2) = tb^j \\ \theta(x_3) = sta^h b^k \end{cases}$$

in the following way:

- If $h \equiv i-1 \mod (3)$ and $k \equiv j-1 \mod (3)$ then θ_0 and θ are conjugated by the element $sta^{-i}b^{-j}$.
- If $h \equiv i+1 \mod (3)$ and $k \equiv j+1 \mod (3)$ then θ_0 and θ are conjugated by the element $a^{-i}b^{-j}$.
- If $h \equiv i-1 \mod (3)$ and $k \equiv j+1 \mod (3)$ then θ_0 and θ are conjugated by the element $sa^{-i}b^{-j}$.
- If $h \equiv i+1 \mod (3)$ and $k \equiv j-1 \mod (3)$ then θ_0 and θ are conjugated by the element $ta^{-i}b^{-j}$.

With the same reasoning one can deal with the other five cases, and so we can represent the six conjugacy classes with

$$\theta_1 : \begin{cases} \theta_1(x_1) = sa \\ \theta_1(x_2) = tb \\ \theta_1(x_3) = stab \end{cases}, \quad \theta_2 : \begin{cases} \theta_2(x_1) = tb \\ \theta_2(x_2) = sa \\ \theta_2(x_3) = stab \end{cases},$$

$$\theta_3 : \begin{cases} \theta_3(x_1) = sa \\ \theta_3(x_2) = stab \\ \theta_3(x_3) = tb \end{cases}, \quad \theta_4 : \begin{cases} \theta_4(x_1) = tb \\ \theta_4(x_2) = stab \\ \theta_4(x_3) = sa \end{cases},$$

$$\theta_5 : \begin{cases} \theta_5(x_1) = stab \\ \theta_5(x_2) = sa \\ \theta_5(x_3) = tb \end{cases}, \quad \theta_6 : \begin{cases} \theta_6(x_1) = stab \\ \theta_6(x_2) = tb \\ \theta_6(x_3) = sa \end{cases}$$

2) *The six classes of epimorphisms fall into 3 classes of epimorphisms.*

By defining the automorphism $\omega : D_3 \times D_3 \to D_3 \times D_3$ by

$$\omega : \begin{cases} \omega(s) = t \\ \omega(t) = s \\ \omega(a) = b \\ \omega(b) = a \end{cases}$$

Then, clearly

$$\theta_1 = \omega \circ \theta_2, \quad \theta_3 = \omega \circ \theta_4, \quad \theta_5 = \omega \circ \theta_6.$$

3) *The remaining three classes are topologically equivalent under the action of $\mathcal{B}$.*

Consider two elements $\phi_{1,2}, \phi_{2,3} \in \mathcal{B}$ defined by

$$\phi_{1,2} : \begin{cases} \phi_{1,2}(x_1) = x_2 \\ \phi_{1,2}(x_2) = x_2^{-1}x_1x_2 \\ \phi_{1,2}(x_3) = x_3 \end{cases} , \quad \phi_{2,3} : \begin{cases} \phi_{2,3}(x_1) = x_1 \\ \phi_{2,3}(x_2) = x_3 \\ \phi_{2,3}(x_3) = x_3^{-1}x_2x_3 \end{cases}$$

Then, $\phi_{2,3} \circ \phi_{1,2}$ takes the epimorphisms of type θ_1 to epimorphisms of type θ_3 and $\phi_{1,2} \circ \phi_{2,3}$ takes epimorphisms of type θ_1 to epimorphisms of type θ_5.

Hence all the epimorphisms are topologically equivalent and defines topologically equivalent actions $D_3 \times D_3$ on the space of trigonal Riemann surfaces with automorphism group $D_3 \times D_3$.
Hence, the space of cyclic trigonal Riemann surfaces of genus 4 determined by Δ_2 are equisymmetric. These surfaces with automorphism group $D_3 \times D_3$ forms a stratum of complex dimension given by $d(\Delta_2) = 0 - 3 + 4 = 1$. This stratum lies in $\mathcal{C}_1$. □

Proposition 2.4.8. *There is one cyclic trigonal Riemann surface Z_4 determined by the Fuchsian group Δ_3 with $s(\Delta_3) = (0; 2, 6, 6)$ and automorphism group $C_6 \times D_3$. $Z_4 \in \mathcal{C}_2$*

Proof. The proof of theorem (2.2.2), where $|G| = 36$, gives yet another existence of an epimorphisms from the Fuchsian groups Δ_5 with signature $s(\Delta_5) = (0; 2, 6, 6)$ onto $C_6 \times D_3$.
The maximal group Δ_5 with signature $(0; 2, 6, 6)$ produces a surface in the family ${}^{18}\mathcal{M}_4^3$ in proposition (2.4.6).
The only possible elevation is $\theta : \Delta_5 \to C_6 \times D_3$ given by, for example,

$$\theta : \begin{cases} \theta(x_1) = s \\ \theta(x_2) = tab \\ \theta(x_3) = stab^2 \end{cases}$$

Note that under conjugation of a the elements of order 2 split up into two conjugacy classes represented by s and st (and one single class containing only t). The possible choice of sending the elements of order 6 is uniquely determined by the two first classes.[29]

$$\theta_1 \begin{cases} \theta_1(x_1) = s \\ \theta_1(x_2) = tab \\ \theta_1(x_3) = stab^2 \end{cases} , \theta_2 \begin{cases} \theta_2(x_1) = st \\ \theta_2(x_2) = tab \\ \theta_2(x_3) = sab^2 \end{cases}$$

In order for θ to be an epimorphism the product of $\theta(x_1)$ and $\theta(x_2)$ will have to end up in one of the conjugacy classes of elements of order 6 represented by $sb^{\pm 1}$, $stb^{\pm 1}$ or $tab^{\pm 1}$.
The element $\phi_{2,3} \in \mathcal{B}$ will interchange the conjugacy classes.

[29] It is not possible to have an epimorphism sending x_1 to t. One can see that such homomorphism would not be surjective.

Since there is only one unique action $(\phi_{2,3}, \omega_a) \in \mathcal{B} \times \mathrm{Aut}\,(C_6 \times D_3)$, taking the epimorphisms into one equivalence class, and the dimension of the strata in this case is $d(\Delta_5) = 0 - 3 + 3 = 0$. There is a unique cyclic trigonal Riemann surface Z_4 of genus 4 with $\mathrm{Aut}\,(Z_4) = C_6 \times D_3$. The surface (stratum) lies in the component $\mathcal{C}_2$.

□

Proposition 2.4.9. *There are exactly 2 cyclic trigonal Riemann surfaces X_4 and Y_4 of genus 4 with automorphism groups of order 72.*

(i) X_4 has one cyclic central trigonal morphism, $Aut(X_4) = \Sigma_4 \times C_3$ and $X_4/\Sigma_4 \times C_3$ uniformized by the Fuchsian group Δ_1 with $s(\Delta_1) = (0; 2, 3, 12)$. $X_4 \in \mathcal{C}_2$

(ii) Y_4 has two trigonal morphisms, $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$ and $Y_4/(C_3 \times C_3) \rtimes D_4$ uniformized by the Fuchsian group Δ_2 with $s(\Delta_2) = (0; 2, 4, 6)$. $Y_4 \in \mathcal{C}_1$

Proof. The proof of theorem (2.2.2), where $|G| = 72$, shows that there are two cases of Fuchsian groups producing cyclic trigonal Riemann surfaces. The first case, with Δ_1 having signature $(0; 2, 3, 12)$, there is a unique epimorphism $\theta_1 : \Lambda_1 \to \Sigma_4 \times C_3$ given by

$$\theta_1 : \begin{cases} \theta_1(x_1) = \overline{s} \\ \theta_1(x_2) = ab \\ \theta_1(x_3) = a^2\overline{s}b \end{cases}$$

The surface given by $X_4 = \mathbb{H}/Ker(\theta)$ with a cyclic central trigonal morphism and with automorphism group $\Sigma_4 \times C_3$ lies in the component $\mathcal{C}_2$. Secondly, the case with Δ_2 having signature $(0; 2, 4, 6)$ the epimorphism $\theta_2 : \Delta_2 \to (C_3 \times C_3) \rtimes D_4$ given by

$$\theta_2 : \begin{cases} \theta_2(x_1) = s \\ \theta_2(x_2) = ta \\ \theta_2(x_3) = stb \end{cases}$$

This epimorphism is an extension of the epimorphisms $\phi_2 : \Lambda_2 \to (C_3 \times C_3) \rtimes C_4$ and $\overline{\phi}_2 : \overline{\Lambda_2} \to D_3 \times D_3$. Where ϕ_2 is given by

$$\phi_2 : \Lambda_2 \to (C_3 \times C_3) \rtimes C_4 \begin{cases} \phi_1(y_1) = a \\ \phi_1(y_2) = at \\ \phi_1(y_3) = bt^3 \end{cases}$$

Now, the braid element $\phi_{2,3} \in \mathcal{B}$ interchanges the conjugacy classes of elements of order 4 in $(C_3 \times C_3) \rtimes C_4$, and so the extensions are topologically equivalent under the action of $(C_3 \times C_3) \rtimes D_4$ and hence there is only one point in the strata, representing the cyclic trigonal Riemann surface $Y_4 = \mathbb{H}/Ker(\theta_2)$ with non-unique trigonal morphism and with automorphism group $(C_3 \times C_3) \rtimes D_4$. □

We gather the information obtained from the propositions (2.4.3 - 2.4.9) into one theorem.

Theorem 2.4.10. *The space $\mathcal{M}_4^3$ of cyclic trigonal Riemann surfaces of genus 4 form a disconnected subspace of dimension 3 of the moduli space $\mathcal{M}_4$. $\mathcal{M}_4^3$ is the union of $\mathcal{C}_1$ and $\mathcal{C}_2$*

1. *The subspace ${}^6\mathcal{M}_4^3$, determined by the Fuchsian groups Δ'' with $s(\Delta'') = (0; 2, 2, 3, 3, 3)$, formed by Riemann surfaces of genus 4 with automorphism group of order 6 is a disconnected space of dimension 2 consisting of two connected components $\mathcal{C}_1^6$, $\mathcal{C}_2^6$ consisting of trigonal Riemann surfaces with automorphism groups D_3 and C_6 respectively. $\mathcal{C}_i^6 \subset \mathcal{C}_i$.*

2. *The subspace ${}^{12}\mathcal{M}_4^3$, determined by the Fuchsian groups Δ' with $s(\Delta') = (0; 2, 2, 3, 6)$, formed by Riemann surfaces of genus 4 with automorphism group of order 12 is a disconnected space of dimension 1 consisting of two connected components $\mathcal{C}_1^{12}$, $\mathcal{C}_2^{12}$ consisting of trigonal Riemann surfaces with automorphism groups D_6 and C_{12} respectively. $\mathcal{C}_i^{12} \subset \mathcal{C}_i$.*

3. *There is one cyclic trigonal Riemann surface T_4 determined by the Fuchsian group Δ_4 with $s(\Delta_4) = (0; 3, 5, 15)$ and automorphism group C_{15}. $T_4 \in \mathcal{C}_2$*

4. *The subspace ${}^{18}\mathcal{M}_4^3$, determined by the Fuchsian groups Δ''' with $s(\Delta''') = (0; 2, 2, 3, 3)$, formed by Riemann surfaces of genus 4 with automorphism group of order 18 is a connected space of dimension 1 consisting of trigonal Riemann surfaces with automorphism group $D_3 \times C_3$. ${}^{18}\mathcal{M}_4^3 \subset \mathcal{C}_2$*

5. *The connected subspace ${}^{36}\mathcal{M}_4^3$ of $\mathcal{M}_4^3$ formed by Riemann surfaces $X_4(\lambda)$ of genus 4 with automorphism group of order 36 is a space of dimension 1 determined by the Fuchsian groups Δ with $s(\Delta) = (0; 2, 2, 2, 3)$. The automorphism group of the Riemann surfaces is $D_3 \times D_3$ and the surfaces admit two trigonal morphisms. ${}^{36}\mathcal{M}_4^3 \subset \mathcal{C}_1$*

6. *There is one cyclic trigonal Riemann surface Z_4 determined by the Fuchsian group Δ_3 with $s(\Delta_3) = (0; 2, 6, 6)$ and automorphism group $C_6 \times D_3$. $Z_4 \in \mathcal{C}_2$*

7. *There are exactly 2 cyclic trigonal Riemann surfaces X_4 and Y_4 of genus 4 with automorphism groups of order 72.*

   [i] X_4 has one cyclic central trigonal morphism, $Aut(X_4) = \Sigma_4 \times C_3$ and $X_4/\Sigma_4 \times C_3$ uniformized by the Fuchsian group Δ_1 with $s(\Delta_1) = (0; 2, 3, 12)$. $X_4 \in \mathcal{C}_2$

   [ii] Y_4 has two trigonal morphisms, $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$ and $Y_4/(C_3 \times C_3) \rtimes D_4$ uniformized by the Fuchsian group Δ_2 with $s(\Delta_2) = (0; 2, 4, 6)$. $Y_4 \in \mathcal{C}_1$

Using coverings and the fundamental groups arising from the Fuchsian groups we can obtain the following relation between the spaces of trigonal Riemann surfaces of genus 4.

Theorem 2.4.11. *The different subspaces in theorem (*2.4.10*) satisfy the following inclusion relations:*

1. *The space* ${}^{12}\mathcal{M}_4^3$ *is a subspace of* ${}^{6}\mathcal{M}_4^3$*. Furthermore* $\mathcal{C}_i^{12} \subset \mathcal{C}_i^{6}$, $i = 1, 2$.
2. *The space* ${}^{18}\mathcal{M}_4^3$ *is a subspace of* $\mathcal{C}_2^6 \subset {}^{6}\mathcal{M}_4^3$.
3. *The space* ${}^{36}\mathcal{M}_4^3$ *is a subspace of* $\mathcal{C}_1^6 \subset {}^{6}\mathcal{M}_4^3$.
4. *The surface* Z_4 *given in theorem (*2.4.10*) belongs to* ${}^{18}\mathcal{M}_4^3 \bigcap \mathcal{C}_2^{12}$.
5. *The surface* X_4 *given in theorem (*2.4.10*) belongs to the space* ${}^{12}\mathcal{M}_4^3$.
6. *The surface* Y_4 *given in theorem (*2.4.10*) belongs to* ${}^{36}\mathcal{M}_4^3 \bigcap \mathcal{C}_1^{12}$.

Proof. 1. In fact the Riemann surfaces $\mathbb{H}/\Delta''$ with $s(\Delta'') = (0; 2, 2, 3, 3, 3)$ are double coverings of the Riemann surfaces $\mathbb{H}/\Delta'$ with $s(\Delta') = (0; 2, 2, 3, 6)$. Consider the map $\phi'' : \Delta' \to \Sigma_2$ defined by

$$\begin{cases} \phi''(x_1') = (1,2) \\ \phi''(x_2') = 1_d \\ \phi''(x_3') = 1_d \\ \phi''(x_4') = (1,2) \end{cases}$$

By theorem (1.3.31), $\phi''(x_1')$ induces no cone points, $\phi''(x_2')$ induces two cone points of order 2, $\phi''(x_3')$ two cone points of order 3 and $\phi''(x_4')$ one of order 3. Thus ϕ'' is the required monodromy of the covering $f_1 : \mathbb{H}/\Delta'' \xrightarrow{2:1} \mathbb{H}/\Delta'$, with $\Delta'' = \phi''^{-1}(Stb(1))$.

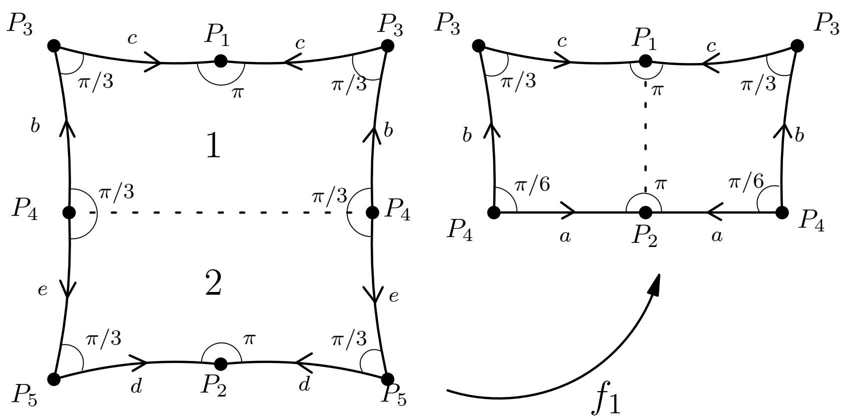

Figure 2.2: The 2-sheeted covering f_1

2. In fact the Riemann surfaces $\mathbb{H}/\Delta''$ with $s(\Delta'') = (0; 2, 2, 3, 3, 3)$ are 3-sheeted coverings of the Riemann surfaces $\mathbb{H}/\Delta'''$ with $s(\Delta''') = (0; 2, 2, 3, 3)$. Consider the map $\phi''' : \Delta''' \to \Sigma_3$ defined by

$$\begin{cases} \phi'''(x_1''') = (1,2) \\ \phi'''(x_2''') = (2,3) \\ \phi'''(x_3''') = 1_d \\ \phi'''(x_4''') = (1,2,3) \end{cases}$$

By theorem (1.3.31), $\phi'''(x_1''')$ and $\phi'''(x_2''')$ induce one cone point of order two each, $\phi'''(x_3''')$ induces three cone points of order 3 and $\phi'''(x_4''')$ none cone point. Thus ϕ''' is the required monodromy of the covering $f_2 : \mathbb{H}/\Delta'' \xrightarrow{3:1} \mathbb{H}/\Delta'''$, with $\Delta'' = \phi'''^{-1}(Stb(1))$. The monodromy ϕ''' yields the action of $C_3 \times D_3$ on the $\langle b, s\rangle$-cosets, where b, s as in case $|G| = 18$ (iv).

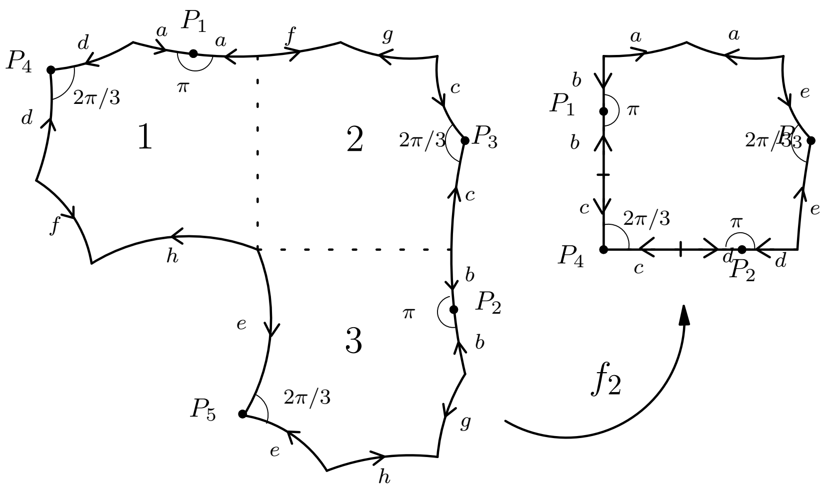

Figure 2.3: The 3-sheeted covering f_2

3. In fact the Riemann surfaces $\mathbb{H}/\Delta''$ with $s(\Delta'') = (0; 2, 2, 3, 3, 3)$ are 6-sheeted coverings of the Riemann surfaces $\mathbb{H}/\Delta$ with $s(\Delta) = (0; 2, 2, 2, 3)$. Consider the maps $\phi_1 : \Delta \to \Sigma_2$ defined by

$$\begin{cases} \phi_1(x_1) = (1,2) \\ \phi_1(x_2) = (1,2) \\ \phi_1(x_3) = 1_d \\ \phi_1(x_4) = 1_d \end{cases}$$

and $\phi_2 : \Lambda \to \Sigma_3$ defined by

$$\begin{cases} \phi_2(y_1) = (1,2) \\ \phi_2(y_2) = (1,3) \\ \phi_2(y_3) = (1,2,3) \\ \phi_2(y_4) = 1_d \end{cases}$$

where $\Lambda = \phi_1^{-1}(Stb(1))$. By theorem (1.3.31) , $s(\Lambda) = (0; 2, 2, 3, 3)$. Now $\phi_2(y_1)$ and $\phi_2(y_2)$ induce one cone point of order 2 each, and $\phi_2(y_4)$ induces three cone points of order 3. Therefore the composition map $\phi_2 \circ \phi_1$ is the required monodromy of the covering $\mathbb{H}/\Delta'' \to \mathbb{H}/\Delta$. Again $\Delta'' = \phi_2^{-1}(Stb(1))$. The monodromy $\phi_2 \circ \phi_1$ yields the action of $D_3 \times D_3$ on the $\langle ab, st\rangle$-cosets, where ab, st are defined in case $|G| = 36$ (ii).

The space ${}^{36}\mathcal{M}_4^3$ belongs to the subvariety $\mathcal{C}_1^6 \subset {}^6\mathcal{M}_4^3$ consisting of the Riemann surfaces with half the stabilizers of the cone points rotating in opposite directions since $\langle ab, st\rangle = D_3$.

Describing the coverings with the coverings of their respective fundamental polygon, the 6-sheeted covering is a composition of the coverings f_2 and $f_3 : \mathbb{H}/\Lambda \xrightarrow{2:1} \mathbb{H}/\Delta$ (given below), such that

$$\mathbb{H}/\Delta'' \xrightarrow{3:1} \mathbb{H}/\Lambda \xrightarrow{2:1} \mathbb{H}/\Delta$$

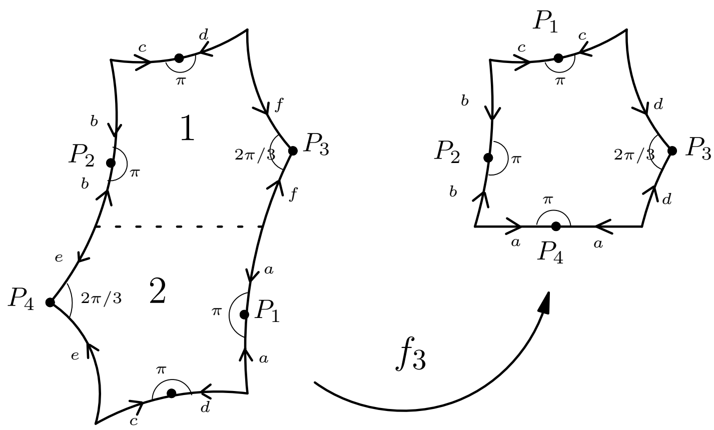

Figure 2.4: The 2-sheeted covering f_3.

4. First of all, the Riemann surfaces $\mathbb{H}/\Delta'$ with $s(\Delta') = (0; 2, 2, 3, 6)$ are 3-sheeted coverings of the Riemann surface $Z_4 = \mathbb{H}/\Delta_3$. Consider the representation $\phi : \Delta_3 \to \Sigma_3$ defined by

$$\begin{cases} \phi(\overline{y}_1) = (1,2) \\ \phi(\overline{y}_2) = (2,3) \\ \phi(\overline{y}_3) = (1,2,3) \end{cases}$$

By theorem (1.3.31), $\phi(\overline{y}_1)$ induces one cone point of order two, $\phi(\overline{y}_2)$ induces one cone point of order 3 and one cone point of order 6 and $\phi(\overline{y}_3)$ induces one cone point of order 2. Then $s(\Delta') = s(\phi^{-1}(Stb(1))) = (0; 2, 2, 3, 6)$. Thus, the map ϕ is the required monodromy of the covering $f_4 : \mathbb{H}/\Delta' \xrightarrow{3:1} \mathbb{H}/\Delta_3$. The monodromy ϕ yields the action of $C_6 \times D_3$ on the $C_6 \times C_2$-cosets, where $C_6 \times C_2 = \langle b, s, t\rangle$ (see case $|G| = 12$ (iii).

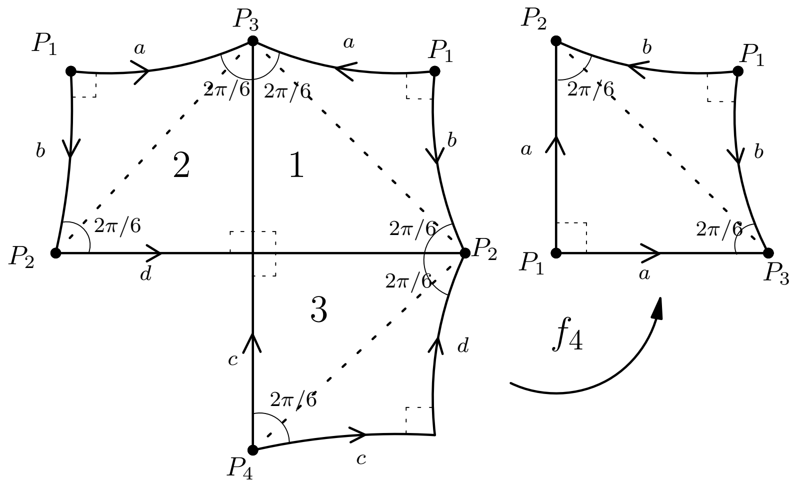

Figure 2.5: The 3-sheeted covering f_4.

Secondly, the Riemann surfaces $\mathbb{H}/\Delta'''$ with $s(\Delta''') = (0; 2, 2, 3, 3)$ are double coverings of the Riemann surface Z_4. By theorem (1.3.31), the map $\tau : \Delta \to \Sigma_2$ defined by

$$\begin{cases} \tau(\overline{x}_1) = 1_d \\ \tau(\overline{x}_2) = (1, 2) \\ \tau(\overline{x}_3) = (1, 2) \end{cases}$$

is the required monodromy of the covering $f_5 : \mathbb{H}/\Delta''' \xrightarrow{2:1} \mathbb{H}/\Delta_3$ with $\Delta = \tau^{-1}(Stb(1))$. Notice that $\tau(\overline{x}_1)$ induces two cone points of order 2, and $\tau(\overline{x}_2)$ and $\tau(\overline{x}_3)$ one cone point of order 3 each. The monodromy τ yields the action of $C_6 \times D_3$ on the $C_3 \times D_3$-cosets, where $C_3 \times D_3 = \langle a, b, s\rangle$ as in case $|G| = 36$ (v).

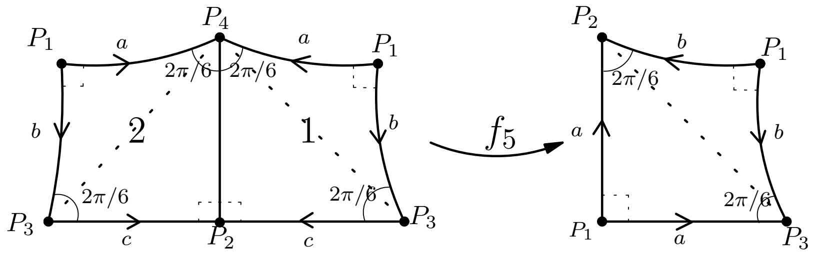

Figure 2.6: The 2-sheeted covering f_5.

5. The Riemann surfaces $\mathbb{H}/\Delta'$ with $s(\Delta') = (0; 2, 2, 3, 6)$ are 6-sheeted coverings of the Riemann surface $\mathbb{H}/\Delta_1$ with $s(\Delta_1) = (0; 2, 3, 12)$.

Consider the representation $\phi : \Delta_1 \to \Sigma_6$ defined by

$$\begin{cases} \phi(\overline{x}_1) = (2,4)(3,5) \\ \phi(\overline{x}_2) = (1,2,3)(4,5,6) \\ \phi(\overline{x}_3) = (1,5,6,2)(3,4) \end{cases}$$

By theorem (1.3.31), $\phi(\overline{x}_1)$ induces two cone points of order 2, $\phi(\overline{x}_2)$ induces no cone points and $\phi(\overline{x}_3)$ induces one cone point of order 3 and one of order 6, then $s(\Delta') = s(\phi^{-1}(Stb(1))) = (0; 2, 2, 3, 6)$. Thus, the map ϕ is the required monodromy of the covering $f_6 : \mathbb{H}/\Delta' \xrightarrow{6:1} \mathbb{H}/\Delta_1$. The monodromy ϕ yields the action of $\Sigma_4 \times C_3$ on the $C_6 \times C_2$-cosets, where $C_6 \times C_2 = \langle b, \overline{s}, (a\overline{s})^2 \rangle$ as in case $|G| = 72$ (i).

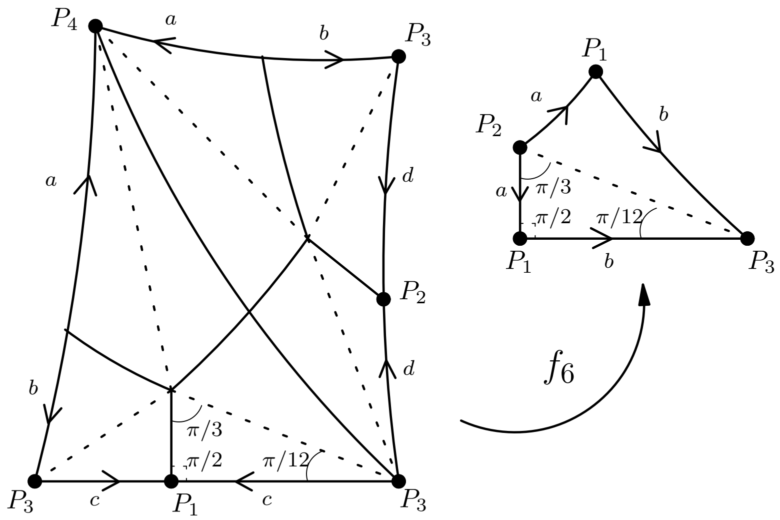

Figure 2.7: The 6-sheeted covering f_6.

6. First of all the Riemann surfaces $\mathbb{H}/\Delta'$ with $s(\Delta') = (0; 2, 2, 3, 6)$ are 6-sheeted coverings of the Riemann surface $Y_4 = \mathbb{H}/\Delta_2$ with $s(\Delta_2) = (0; 2, 4, 6)$.

   Consider the representation $\phi : \Delta_2 \to \Sigma_6$ defined by

$$\begin{cases} \phi(\overline{y}_1) = (1,4)(2,3)(5,6) \\ \phi(\overline{y}_2) = (2,4,5,6)(1,3) \\ \phi(\overline{y}_3) = (1,2,5)(3,4) \end{cases}$$

   By theorem (1.3.31), $\phi(\overline{y}_1)$ induces no cone points, $\phi(\overline{y}_2)$ induces one cone point of order 2 and $\phi(\overline{y}_3)$ induces one cone point of order 3, one cone point of order 6 and one of order 2, then $s(\Delta') = s(\phi^{-1}(Stb(1))) = (0; 2, 2, 3, 6)$. Thus, the map ϕ is the required monodromy of the covering $f_7 : \mathbb{H}/\Delta' \xrightarrow{6:1} \mathbb{H}/\Delta_2$. The monodromy ϕ yields the action of $(C_3 \times C_3) \rtimes D_4$ on the D_6-cosets, where $D_6 = \langle a, s, t^2 \rangle$ as in case $|G| = 72$ (ii).

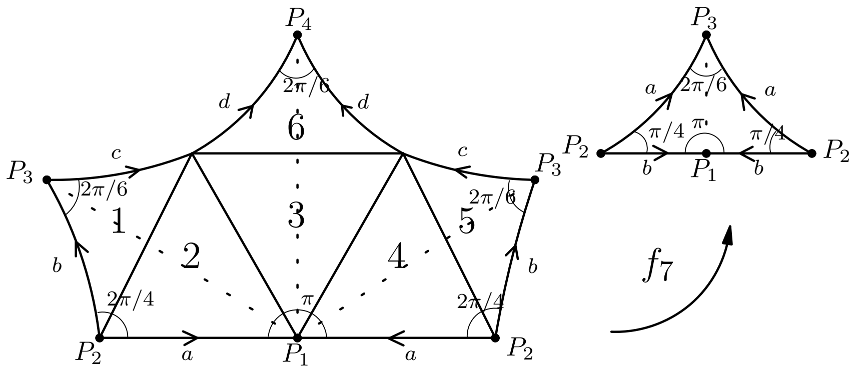

Figure 2.8: The 6-sheeted covering f_7.

Secondly, the Riemann surfaces $\mathbb{H}/\Delta$ with $s(\Delta) = (0; 2, 2, 2, 3)$ are double coverings of the Riemann surface $Y_4 = \mathbb{H}/\Delta_2$ with $s(\Delta_2) = (0; 2, 4, 6)$. By theorem (1.3.31), the map $\tau : \Delta \to \Sigma_2$ defined by

$$\begin{cases} \tau(\overline{x}_1) = 1_d \\ \tau(\overline{x}_2) = (1, 2) \\ \tau(\overline{x}_3) = (1, 2) \end{cases}$$

is the required monodromy of the covering $f_8 : \mathbb{H}/\Delta \xrightarrow{2:1} \mathbb{H}/\Delta_2$ with $\Delta = \tau^{-1}(Stb(1))$. Notice that $\tau(\overline{x}_1)$ induces two cone points of order 2, $\tau(\overline{x}_2)$ the third cone point of order 2 and $\tau(\overline{x}_3)$ one cone point of order 3. The monodromy τ yields the action of $(C_3 \times C_3) \rtimes D_4$ on the $D_3 \times D_3$-cosets, where $D_3 \times D_3 = \langle a, b, s, t^2 \rangle$ as in case $|G| = 72$ (ii).

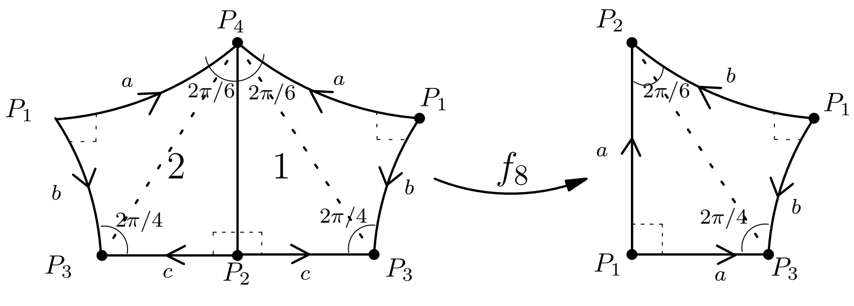

Figure 2.9: The 2-sheeted covering f_8.

□

Remark 2.4.12. *The Riemann surfaces* $\mathbb{H}/\Delta'$ *with* $s(\Delta') = (0; 2, 2, 3, 3, 3)$ *cannot be coverings of the Riemann surfaces* $\mathbb{H}/\Delta$ *with* $s(\Delta) = (0; 2, 2, 2, 3)$. *Hence the space* ${}^{36}\mathcal{M}_4^3$ *is not a subspace of* $\mathcal{C}_2^{12} \subset {}^{12}\mathcal{M}_4^3$.

Remark 2.4.13. *The surface* X_4 *does not belong to* ${}^{18}\mathcal{M}_4^3$ *since the group* $C_3 \times D_3$ *in not a subgroup of the group* $\Sigma_4 \times C_3$.

2.5 The space of trigonal Riemann surfaces with non-unique trigonal morphisms

The following results appear in

[14] A. Costa, M. Izquierdo, D. Ying, *On the family of cyclic trigonal Riemann surfaces of genus 4 with several trigonal morphisms*, Submitted 2006.

The result of proposition (2.4.7) can be extended further and it is possible to identify the moduli space ${}^{36}\mathcal{M}_4^3$ as a Riemann surface.

Theorem 2.5.1. *The space* ${}^{36}\mathcal{M}_4^3$ *is a Riemann surface. Furthermore, it is the Riemann sphere with three punctures.*

Proof. According to proposition (2.4.7), ${}^{36}\mathcal{M}_4^3$ is an equisymmetric strata of complex dimension 1. This means that it is connected and hence is a Riemann surface. The space ${}^{36}\mathcal{M}_4^3$ can be identified with the moduli space of orbifolds $\mathbb{H}/\Delta$, $s(\Delta) = (0; 2, 2, 2, 3)$, having three cone points of order 2 and one of order 3.
Since the automorphism group of the surfaces in ${}^{36}\mathcal{M}_4^3$ is $D_3 \times D_3$, we can represent the cone points of order 2 by the conjugacy classes of involutions in $D_3 \times D_3$. Recall that $D_3 \times D_3$ is generated by a, b, s, t, where the last two have order 2. The three classes are represented by $[s]$, $[t]$ and $[st]$.
Now, using a Möbius transformation we can assume that the three order 2 cone points are 0, 1 and ∞, each corresponding to $[s]$, $[t]$ and $[st]$ respectively.
If we let the cone point of order 3 be represented by λ then ${}^{36}\mathcal{M}_4^3$ is parameterized by the position of λ and the map

$$\Phi : {}^{36}\mathcal{M}_4^3 \to \widehat{\mathbb{C}} \backslash \{0, 1, \infty\}$$

sending $X(\lambda)$ to λ is an isomorphism. That is

$${}^{36}\mathcal{M}_4^3 \cong \widehat{\mathbb{C}} \backslash \{0, 1, \infty\}$$

□

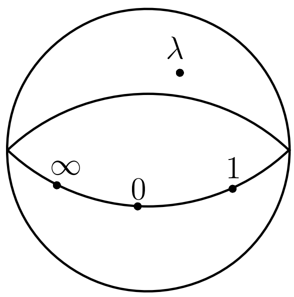

Figure 2.10: The Riemann sphere without 0, 1 and ∞

Chapter 3

On cyclic p-gonal Riemann surfaces with several p-gonal morphisms

The following results appear in

[15] A. Costa, M. Izquierdo, D. Ying, *On cyclic p-gonal Riemann surfaces with several p-gonal morphisms*, Submitted 2006.

From the results of section 2.3 and section 2.5 it is natural to look for a generalization. In this chapter the result will be extended to cyclic p-gonal Riemann surfaces and this will show that Accola's bound given in the lemma (2.1.3 (i))

3.1 p-gonal Riemann surfaces

In the following, let $p > 2$ be a prime number. A Riemann surface X is said to be *p-gonal* if it admits a p-sheeted covering $f : X \to \widehat{\mathbb{C}}$ onto the Riemann sphere. If f is a cyclic regular covering then X is called cyclic p-gonal. The covering f will be called the (cyclic) p-gonal morphism. The equation for cyclic p-gonal surfaces is given by

$$y^p = p(x).$$

The following result gives us a characterization of cyclic p-gonal Riemann surfaces using Fuchsian groups. See [11] (See [12] for the case $p = 3$):

Theorem 3.1.1. *Let p be a prime number. Let X_g be a Riemann surface, X_g admits a cyclic p-gonal morphism f if and only if there is a Fuchsian* *group* Δ *with signature* $(0;\overbrace{p,\ldots,p}^{\frac{2g}{p-1}+2})$ *and an index* p *normal surface subgroup* Γ *of* Δ*, such that* Γ *uniformizes* X_g*.*

Proof. Let (X,f) be a cyclic p-gonal surface. Then there is an order p automorphism $\varphi : X \to X$ such that $X/\langle\varphi\rangle$ is the Riemann sphere and φ is a deck-transformation of the covering map f. Let Γ be a Fuchsian surface group uniformizing X and let $\widetilde{\varphi}$ be a lifting of φ to the universal covering $\mathbb{H} \to \mathbb{H}/\Gamma = X$. We call $\Delta = \langle\Gamma,\widetilde{\varphi}\rangle$ the universal covering transformations group of $(X,\langle\varphi\rangle)$. Since $f : \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$ is a p-sheeted regular covering, by the Riemann-Hurwitz formula the signature of Δ is $(0;\overbrace{p,...,p}^{\frac{2g}{p-1}+2})$ and Γ is an index p normal surface subgroup of Δ.

Now, let Δ be a Fuchsian group with signature $(0;\overbrace{p,...,p}^{\frac{2g}{p-1}+2})$ and Γ an index p normal surface subgroup of Δ, such that $X = \mathcal{D}/\Gamma$. Then $f : X = \mathcal{D}/\Gamma \to \mathcal{D}/\Delta$ is a cyclic p-gonal morphism and (X,f) is a cyclic p-gonal surface. $\square$

By Lemma 2.1 in [1], if the surface X_g has genus $g \geq (p-1)^2+1$, then the p-gonal morphism is unique. In this case, the cyclic p-gonal morphism f is induced by a normal subgroup C_p in $Aut(X_g)$. See [18]. The methods in [1] can not be applied for small genera.
There are many well-known examples of p-gonal Riemann surfaces with non-unique $p$gonal morphisms. First of all, in the case of $p = 3$ we have seen in chapter 2 that there exist cyclic trigonal Riemann surfaces of genus 4 having two trigonal morphisms.
Another example is the Klein quartic, which is a 7-gonal Riemann surface having automorphism group $PSL_2(7)$ and where C_7 is non-normal in $PSL_2(7)$
We use Theorem (3.1.1) to obtain the following algorithm to find the spaces ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ of cyclic p-gonal Riemann surfaces of genera $(p-1)^2$ with several p-gonal morphisms and thus proving that the bound above is sharp.

Algorithm for finding spaces ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$

Let $G = Aut(X_g)$, with $g = (p-1)^2$, and let $X_g = \mathbb{H}/\Gamma$ be a Riemann surface of genus $g = (p-1)^2$ uniformized by the surface Fuchsian group Γ. The surface X_g admits a cyclic p-gonal morphism f if and only if there is a maximal Fuchsian group Δ with signature $(0;m_1,...,m_r)$, an order p automorphism $\varphi : X_g \to X_g$, such that $\langle\varphi\rangle \leq G$ and an epimorphism $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$ in such a way that $\theta^{-1}(\langle\varphi\rangle)$ is a Fuchsian group with signature $(0;\overbrace{p,...,p}^{\frac{2(p-1)^2}{p-1}+2}) = (0;\overbrace{p,...,p}^{2p})$. Furthermore the p-gonal morphism f is unique if and only if $\langle\varphi\rangle$ is **normal** in G (see [18]).

Since we assume that there are at least two p-gonal morphism, we consider the groups $G = D_p \times D_p$, which contain two conjugated subgroups of order p. See [14] for $p = 3$.

Remark 3.1.2. *It is interesting to enumerate the conjugacy classes of subgroups of order 2 and p in the group* $G = D_p \times D_p = \langle a, b, s, t / a^p = b^p = s^2 = t^2 = [a, b] = [s, b] = [t, a] = (sa)^2 = (tb)^2 = (st)^2 = 1\rangle$. *The group* $D_p \times D_p$ *contains the following conjugacy classes of subgroups of order 2 and p:*

1. *three conjugacy classes of involutions:* $\langle sa^i\rangle$, $\langle tb^j\rangle$ *and* $\langle sta^ib^j\rangle$,
2. *two conjugacy classes of normal subgroups of order p:* $\langle a\rangle$ *and* $\langle b\rangle$,
3. $\frac{p-1}{2}$ *conjugacy classes of subgroups of order p:* $\langle a^ib\rangle$, $i \in \{1, 2, \ldots, p-1\}$. *Observe that the subgroup H generated by* a^ib^j *is conjugated to the subgroup* H' *generated by* $a^{-i}b^j$. *Notice also that the subgroups* H' *and* H'' *generated by* $a^{-i}b^j$ *and* a^ib^{-j} *respectively coincide.*

Theorem 3.1.3. *There is a uniparametric family* ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ *of Riemann surfaces* $X_{(p-1)^2}(\lambda)$ *of genus* $(p-1)^2$ *admitting two cyclic p-gonal morphisms. The surfaces* $X_{(p-1)^2}(\lambda)$ *have* $Aut(X_{(p-1)^2}(\lambda)) = D_p \times D_p$ *as automorphism group and the quotient Riemann surfaces* $X_{(p-1)^2}(\lambda)/G$ *are uniformized by the Fuchsian groups* Δ *with signature* $s(\Delta) = (0; 2, 2, 2, p)$.

Proof. Consider the finite group $G = D_p \times D_p$. By the algorithm above and the Riemann-Hurwitz formula G is the automorphism group of surfaces of genus $(p-1)^2$ if there is an epimorphism from the Fuchsian groups Δ with signature $(0; 2, 2, 2, p)$ onto G.
Now, consider the epimorphism $\theta : \Delta \to D_p \times D_p$ defined by

$$\theta : \begin{cases} \theta(x_1) = s \\ \theta(x_2) = t \\ \theta(x_3) = stab \\ \theta(x_4) = a^{p-1}b^{p-1} \end{cases}$$

The coset tables for $\langle ab\rangle$ have representatives

$$\begin{array}{l}\{[1], [a], [a^2], \ldots, [a^{p-1}], [s], [sa], [sa^2], \ldots, [sa^{p-1}],\\ [t], [ta], [ta^2], \ldots, [ta^{p-1}], [st], [sta], [sta^2], \ldots, [sta^{p-1}]\}\end{array}$$

and the coset tables for $\langle a^{p-1}b^{p-1}$ have representatives

$$\begin{array}{l}\{[1], [a], [b], [a^2b], \ldots, [a^{p-1}b], [s], [sa], [sb][sa^2b], \ldots, [sa^{p-1}b],\\ [t], [ta], [tb][ta^2b], \ldots, [ta^{p-1}b], [st], [sta], [stb], [sta^2b], \ldots, [sta^{p-1}b]\}\end{array}$$

The action of $\theta(x_4) = a^{p-1}b^{p-1}$ on the $\langle ab\rangle$-cosets has the following orbits:

$$\begin{array}{l}\{[1], [a], [b], [a^2b], \ldots, [a^{p-1}b]\},\\ \{[s]\}, \{[sa]\}, \{[sb]\}, \{[sa^2b]\}, \ldots, \{[sa^{p-1}b]\},\\ \{[t]\}, \{[ta]\}\ \{[tb]\}, \{[ta^2b]\}, \ldots, \{[ta^{p-1}b]\},\\ \{[st], [sta], [stb], [sta^2b], \ldots, [sta^{p-1}b]\}.\end{array}$$

Then $s(\theta^{-1}(\langle ab\rangle))$ contains $2p$ periods of order p and by the Riemann-Hurwitz formula $s(\theta^{-1}(\langle ab\rangle)) = (0;\overbrace{p,...,p}^{2p})$.
In the same way the action of $\theta(x_4) = a^{p-1}b^{p-1}$ on the $\langle a^{p-1}b^{p-1}\rangle$-cosets has the orbits:

$$\begin{array}{l}\{[1]\},\{[a]\},\{[b]\},\{[a^2b]\},\ldots,\{[a^{p-1}b]\},\\ \{[s],[sa]\,[sb],[sa^2b],\ldots,[sa^{p-1}b]\},\\ \{[t],[ta]\,[tb],[ta^2b],\ldots,[ta^{p-1}b]\},\\ \{[st]\},\{[sta]\},\{[stb]\},\{[sta^2b]\},\ldots,\{[sta^{p-1}b]\}\end{array}$$

Again, the signature $s(\theta^{-1}(\langle a^{p-1}b^{p-1}\rangle))$ contains $2p$ periods equal to p and then $s(\theta^{-1}(\langle a^{p-1}b^{p-1}\rangle)) = (0;\overbrace{p,...,p}^{2p})$. Thus the Riemann surfaces uniformized by $Ker(\theta)$ are cyclic p-gonal Riemann surfaces that admit two different trigonal morphisms $f_1 : \mathbb{H}/Ker(\theta) \to \widehat{\mathbb{C}}$ and $f_2 : \mathbb{H}/Ker(\theta) \to \widehat{\mathbb{C}}$ induced by the subgroups $\langle ab\rangle$ and $\langle a^{p-1}b^{p-1}\rangle$ of $D_p \times D_p$. The dimension of the family of surfaces $\mathbb{H}/Ker(\theta)$ is given by the dimension of the space of groups Δ with $s(\Delta) = (0;2,2,2,p)$. This (complex-)dimension is $3(0) - 3 + 4 = 1$.
Note that if H' is a subgroup of order p in $G = D_p \times D_p$ not conjugated to $H = \langle a^ib^j\rangle$, then the action of a^ib^j on the H'-cosets has no fixed points. $\square$

3.2 Actions of finite groups on p-gonal Riemann surfaces

Our aim is to show that the spaces ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ are connected and hence Riemann surfaces. To do that we will prove, by means of Fuchsian groups, that there is exactly one class of actions of $D_p \times D_p$ on the surfaces $X_{(p-1)^2}(\lambda)$.
Each *(effective and orientable) action* of $G = D_p \times D_p$ on a surface $X = X_{(p-1)^2}(\lambda)$ is determined by an epimorphism $\theta : \Delta \to G$ from the Fuchsian group Δ such that $ker(\theta) = \Gamma$, where $X_{(p-1)^2}(\lambda) = \mathbb{H}/\Gamma$ and Γ is a surface Fuchsian group. The group Δ has signature $s(\Delta) = (0;2,2,2,p)$ and presentation $\langle x_1, x_2, x_3, x_4\ x_1^2 = x_2^2 = x_3^2 = x_4^p = x_1x_2x_3x_4 = 1\rangle$.

Remark 3.2.1. *The condition $X_{(p-1)^2}(\lambda) = \mathbb{H}/\Gamma$ with Γ a surface Fuchsian group imposes:*

1. $o(\theta(x_1)) = o(\theta(x_2)) = o(\theta(x_3)) = 2$,
2. $o(\theta(x_4)) = p$ *and*
3. $\theta(x_1)\theta(x_2)\theta(x_3) = \theta(x_4)^{-1}$.

Two actions $\varepsilon, \varepsilon'$ of G on a Riemann surface X are *(weakly) topologically equivalent* if there is an $w \in Aut(G)$ and an $h \in Hom^+(X)$ such that $\varepsilon'(g) = h\varepsilon w(g)h^{-1}$.

Paraphrasing it in terms of groups: two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define two topologically equivalent actions of G on X if there exist automorphisms $\phi : \Delta \to \Delta$, $w : G \to G$ such that $\theta_2 = w \cdot \theta_1 \cdot \phi^{-1}$.
With other words, let $\mathcal{B}$ be the subgroup of $Aut(\Delta)$ induced by orientation preserving homeomorphisms. Then two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define the same class of G-actions if and only if they lie in the same $\mathcal{B} \times Aut(G)$-class. See [8].
We need now an algebraic characterization of $\mathcal{B}$. Consider the group

$$\overline{\Delta} = \langle \overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4 : \ \overline{x}_1\overline{x}_2\overline{x}_3\overline{x}_4 = 1 \rangle \tag{3.1}$$

$\overline{\Delta}$ is the fundamental group of the punctured surface X_0 obtained by removing the four branch points of the quotient Riemann sphere X/G. $\mathcal{B}$ can be identified with a certain subgroup of the mapping class group of X_0.
Now, any automorphism $\phi \in \Delta$ can be lifted to an automorphism $\overline{\phi} \in \overline{\Delta}$ such that for $1 \leq j \leq 4$ $\overline{\phi}(\overline{x}_j)$ is conjugate to some $(\overline{x}_j)$. The induced representation $\mathcal{B} \to \Sigma_4$ preserves the branching orders.
We are interested in finding elements of $\mathcal{B} \times Aut(G)$ that make our epimorphisms $\theta_1, \theta_2 : \Delta \to G$ equivalent. We can produce the automorphism $\phi \in \mathcal{B}$ ad hoc. In our case the only elements $\mathcal{B}$ we need are compositions of $x_j \to x_{j+1}$ and $x_{j+1} \to x_{j+1}^{-1} x_j x_{j+1}$, where we write down only the action on the generators moved by the automorphism. See [8]. Of course the induced representation above induced by these movements must preserve the branching orders.
We recall again $G = D_p \times D_p = \langle a, b, s, t,\ a^p = b^p = s^2 = t^2 = (st)^2 = [a, b] = (sa)^2 = (tb)^2 = [s, b] = [t, a] = 1 \rangle$.

Lemma 3.2.2. *There is an epimorphism $\theta : \Delta \to G$ satisfying the Remark 4 if and only if $\theta(x_4) = a^{\varepsilon} b^{\delta}$, where $\varepsilon, \delta \in \{1, 2, \ldots, p-1\}$.*

Proof. The elements of order three in G are $a^{\varepsilon} b^{\delta}$, a^i and b^j, $i, j \in \{1, 2, \ldots, p-1\}$. If $\theta(x_4) = a^i$ or $\theta(x_4) = b^j$ then the action of $\theta(x_4)$ on the $\langle a \rangle$- and $\langle b \rangle$-cosets leaves $4p$ fixed cosets which is geometrically impossible. □

Using Lemma 5 and Remark 4 we obtain all the epimorphisms $\theta : \Delta \to G$. We list them in 6 cases depending on conjugacy classes of involutions of G. They are defined as follows:

1. $\theta(x_1) = sa^i, \quad \theta(x_2) = tb^j, \quad \theta(x_3) = sta^h b^k$
2. $\theta(x_1) = tb^j, \quad \theta(x_2) = sa^i, \quad \theta(x_3) = sta^h b^k$
3. $\theta(x_1) = tb^j, \quad \theta(x_2) = sta^i b^k, \quad \theta(x_3) = sa^h$
4. $\theta(x_1) = sa^i, \quad \theta(x_2) = sta^h b^j, \quad \theta(x_3) = tb^k$
5. $\theta(x_1) = sta^i b^j, \quad \theta(x_2) = tb^k, \quad \theta(x_3) = sa^h$
6. $\theta(x_1) = sta^i b^j, \quad \theta(x_2) = sa^h, \quad \theta(x_3) = tb^k$

where $0 \leq i \leq p$, $0 \leq j \leq p$, $i \not\equiv h \mod (p)$ and $j \not\equiv k \mod (p)$.

Lemma 3.2.3. *All the epimorphisms within the same case define the same action of the group G on the Riemann surface X.*

Proof. In Case 1. It is enough to show that the epimorphism

$$\theta_0 : \begin{cases} \theta_0(x_1) = s \\ \theta_0(x_2) = t \\ \theta_0(x_3) = stab \end{cases}$$

is conjugated to an epimorphism

$$\theta : \begin{cases} \theta(x_1) = sa^i \\ \theta(x_2) = tb^j \\ \theta(x_3) = sta^h b^k \end{cases},$$

where $0 \leq i \leq p-1$, $0 \leq j \leq p-1$, $i \not\equiv h \mod (p)$ and $j \not\equiv k \mod (p)$. We consider the intermediate epimorphisms θ_1 defined by

$$\theta_1 : \begin{cases} \theta_1(x_1) = s \\ \theta_1(x_2) = t \\ \theta_1(x_3) = sta^{h-i} b^{k-j} \end{cases},$$

with $h - i \neq 0 \mod (p)$ and $k - j \neq 0 \mod (p)$.
First of all, the epimorphism θ is conjugated to the corresponding epimorphism θ_1 by the element $sta^{i'} b^{j'}$ of G if $2i' \equiv i \mod (p)$ and $2j' \equiv j \mod (p)$.
Now, the epimorphisms θ_1 and θ_0 define the same action of G. Indeed $1_d \times w_{\frac{1}{h-i},\frac{1}{h-i}} \in \mathcal{B} \times Aut(G)$, where the automorphism $w_{\frac{1}{h-i},\frac{1}{h-i}} : G \to G$ is defined by

$$w_{\frac{1}{h-i},\frac{1}{h-i}} : \begin{cases} w_{\frac{1}{h-i},\frac{1}{h-i}}(s) = s \\ w_{\frac{1}{h-i},\frac{1}{h-i}}(t) = t \\ w_{\frac{1}{h-i},\frac{1}{h-i}}(a) = a^x \\ w_{\frac{1}{h-i},\frac{1}{h-i}}(b) = b^y \end{cases},$$

where x and y satisfy the equations $(h-i)x \equiv 1 \mod (p)$, $(k-j)y \equiv 1 \mod (p)$, commutes the epimorphism θ_1 with the epimorphisms θ_0.
The reasoning is similar in all the others cases. $\square$

Lemma 3.2.4. *Epimorphisms of type 1 and 2 define the same action of $G = D_p \times D_p$ on X. Epimorphisms of type 3 and 4 define the same action of $G = D_p \times D_p$ on X. Finally, epimorphisms of type 5 and 6 define the same action of $G = D_p \times D_p$ on X.*

Proof. In fact, $1_d \times w \in \mathcal{B} \times Aut(G)$, where the automorphism $w : G \to G$ is defined by

$$w : \begin{cases} w(s) = t \\ w(t) = s \\ w(a) = b \\ w(b) = a \end{cases},$$

commutes epimorphisms of type 1 with epimorphisms of type 2; the automorphism w commutes epimorphisms of type 3 with epimorphisms of type 4; finally $1_d \times w$ commutes also epimorphisms of type 5 with epimorphisms of type 6. □

We have seen that there are three actions (up to automorphisms) of the group G on the surfaces X. We now show that the actions are topologically equivalent under the action of $\mathcal{B}$.

Theorem 3.2.5. *There is a unique class of actions of the finite group $G = D_p \times D_p$ on the surfaces $X = X_{(p-1)^2}(\lambda)$.*

Proof. By Lemmas 4, 5 and 6 it is enough to show that there are elements of $\mathcal{B}$ commuting an epimorphism of type 1 (or 2) with some epimorphism of type 3 (4), and an epimorphism of type 1 (2) with some epimorphism of type 5 (6).

Consider $\phi_{1,2} : \Delta \to \Delta$ and $\phi_{2,3} : \Delta \to \Delta$ defined by

$$\phi_{1,2} : \begin{cases} \phi_{1,2}(x_1) = x_2 \\ \phi_{1,2}(x_2) = x_2^{-1}x_1x_2 \\ \phi_{1,2}(x_3) = x_3 \end{cases}, \phi_{2,3} : \begin{cases} \phi_{2,3}(x_1) = x_1 \\ \phi_{2,3}(x_2) = x_3 \\ \phi_{2,3}(x_3) = x_3^{-1}x_2x_3 \end{cases}$$

In the first case $\phi_{2,3} \cdot \phi_{1,2}$ takes the epimorphism type 1.

$$\theta_0 : \begin{cases} \theta_0(x_1) = s \\ \theta_0(x_2) = t \\ \theta_0(x_3) = stab \end{cases}$$

to the epimorphism

$$\theta_1 : \begin{cases} \theta_1(x_1) = t \\ \theta_1(x_2) = stab \\ \theta_1(x_3) = sa^{p-1} \end{cases}$$

of type 3.

In the second case $\phi_{1,2} \cdot \phi_{2,3}$ takes the epimorphism type 1.

$$\theta_0 : \begin{cases} \theta_0(x_1) = s \\ \theta_0(x_2) = t \\ \theta_0(x_3) = stab \end{cases}$$

to the epimorphism

$$\theta_2 : \begin{cases} \theta_2(x_1) = stab \\ \theta_2(x_2) = sa^2 \\ \theta_2(x_3) = tb^{p-1} \end{cases}$$

of type 6. □

As a consequence of the previous theorem we obtain

Theorem 3.2.6. *The spaces* ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ *are Riemann surfaces. Furthermore each of them is the Riemann sphere with three punctures.*

Proof. By Theorem 8, each ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ is a connected space of complex dimension 1. Each space ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ can be identified with the moduli space of orbifolds with three cone points of order 2 and one of order 3. Each cone point of order 2 corresponds to a conjugacy class of involutions in $D_p \times D_p : [s], [t]$ and $[st]$. Using a Möbius transformation we can assume that the three order two cone points are 0 (corresponding to $[s]), 1$ (corresponding to $[t]$) and ∞ (corresponding to $[st]$). Thus each ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ is parameterized by the position λ of the order three cone point and the map $\Phi :{}^{4p^2} \mathcal{M}^p_{(p-1)^2} \ni X(\lambda) \to \lambda \in \widehat{\mathbb{C}} - \{0, 1, \infty\}$ is an isomorphism. Hence each ${}^{4p^2}\mathcal{M}^p_{(p-1)^2}$ is the Riemann sphere with three punctures. $\square$

Chapter 4

Conclusions and further work

This thesis completely classifies the cyclic trigonal Riemann surfaces of genus 4. It shows which Riemann surfaces of genus 4 that have non-unique trigonal morphisms and we emphasize particularly which these surfaces are.

Moreover, the space of cyclic trigonal Riemann surfaces of genus 4 is studied and the strata of cyclic trigonal Riemann surfaces of genus 4 classified according to their automorphism groups are also studied. Again, we emphasize the space of cyclic trigonal Riemann surfaces of genus 4 having several trigonal morphisms.

In fact, the cyclic trigonal Riemann surfaces of genus 2,3 and 4 have now been completely studied. The cases of genus 2 and 3 was studied in [13].

When extending the result to the general cyclic p-gonal Riemann surfaces, a natural question might be what are the cyclic p-gonal Riemann surfaces of genus lower than $(p-1)^2$? What does their space look like?

In the general case, we are dealing with p-gonal morphisms and in this case the calculations become more difficult. However, the methods in this thesis should be applicable for this problem, but the question remains still open.

Recall lemma 2.1.3 from [1]. This lemma gives a lower bound, for the genus of a surface, in order for that surface to have a unique p-gonal morphism. That is, if $X_g \to X_q$ is a p-sheeted covering then if p is prime and

$$g > 2pq + (p-1)^2$$

then there is only one such covering with the given p and q. Thus, one can also work with coverings of surfaces of genera $q > 0$ and again ask the same question: Which Riemann surfaces of genus g admit several coverings (morphisms) if $g \leq 2pq + (p-1)^2$?

Recently, Accola (2006) found a Riemann surface W_{10} of genus 10 admitting 4 coverings of a tori, each in three sheets [4]. Instead of working with coverings of the sphere he uses coverings of the tori. The research in this area is very active today.

In section 2.5 we showed that the space of trigonal Riemann surfaces with non-unique trigonal morphisms (${}^{36}\mathcal{M}_4^3$) is identical with the Riemann sphere without 3 points. Another question might also be what the spaces of the trigonal Riemann surfaces are in the other cases, i.e. for ${}^{6}\mathcal{M}_4^3, {}^{12}\mathcal{M}_4^3$ and ${}^{18}\mathcal{M}_4^3$. Note that the first two are not connected spaces, they are in fact the disjoint union of two Riemann surfaces.

Appendix A

List of groups

We follow Coxeter's and Moser's notation for groups up to order 27.

$Q = \langle s, t | s^4 = t^4 = 1, s^2 = t^2 \rangle$

Elements

Order	Elements	Number of elements
2	$s^2 (= t^2)$	1
4	$s^{\pm 1}$, $t^{\pm 1}$, $s^{\pm 1} t$	6
Total number of elements		8

Groups of order 12

1. $C_{12} = \langle a | a^{12} = 1 \rangle$

2. $C_6 \times C_2 = \langle a, s | a^6 = s^2 = [a, s] = 1 \rangle$

3. $D_6 = \langle a, s | a^6 = s^2 = (sa)^2 = 1 \rangle$

4. $A_4 = \langle a, s | a^3 = s^2 = (as)^3 = 1 \rangle$

Elements

Order	Elements	Number of elements
2	s, asa^2, a^2sa	3
3	$a^{\pm 1}$, $sa^{\pm 1}$, $a^{\pm 1}s$, asa, sas	8
Total number of elements		12

5. $\langle 2, 2, 3 \rangle = \langle a, b | a^3 = b^4 = 1, b^3ab = a^2 \rangle = T$

Elements

Order	Elements	Number of elements
2	b^2	1
3	$a^{\pm 1}$	2
4	$b^{\pm 1}$, $a^{\pm 1}b^{\pm 1}$	6
6	$a^{\pm 1}s^2$	2
Total number of elements		12

Groups of order 15

$C_{15} = \langle a | a^{15} = 1 \rangle$

Groups of order 18

1. $C_{18} = \langle a | a^{18} = 1 \rangle$
2. $C_6 \times C_3 = \langle a, b, t | a^3 = b^3 = t^2 = [a, b] = [a, t] = [b, t] = 1 \rangle$
3. $C_3 \times D_3 = \langle a, b, s | a^3 = b^3 = s^2 = [a, b] = [s, b] = (sa)^2 = 1 \rangle$

Elements

Order	Elements	Number of elements
2	sa^i $(i = 0, 1, 2)$	3
3	$a^i b^j$	8
6	$sb^{\pm 1}$, $sa^{\pm 1}b^{\pm 1}$	6
Total number of elements		18

4. $D_9 = \langle a, s | a^9 = s^2 = (sa)^2 = 1 \rangle$
5. $\langle 3, 3, 3, 2 \rangle = \langle a, b, s | a^3 = b^3 = s^2 = (sa)^2 = (sb)^2 = [a, b] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	$sa^i b^j$	9
3	$a^i b^j$, $(i, j) \neq (0, 0)$	8
Total number of elements		18

Groups of order 21

1. $C_{21} = \langle a | a^{21} = 1 \rangle$
2. $C_7 \rtimes C_3 = \langle a, b | a^3 = b^7 = 1, a^2ba = b^4 \rangle$

Elements

Order	Elements	Number of elements
3	$a^{\pm 1}b^i$, $(i = 0, \ldots, 6)$	14
7	b^i, $(i = 1, \ldots, 6)$	6
Total number of elements		21

Groups of order 24

1. $C_{24} = \langle a | a^{24} = 1 \rangle$
2. $C_2 \times C_{12} = \langle a, s | a^{12} = s^2 = [a, s] = 1 \rangle$
3. $C_2 \times C_2 \times C_6 = \langle a, s, t | a^6 = s^2 = t^2 = [a, s] = [a, t] = [s, t] = 1 \rangle$
4. $C_2 \times A_4 = \langle a, s, t | a^3 = s^2 = t^2 = (as)^3 = [a, t] = [s, t] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	s, asa^2, a^2sa t, st, asa^2t, a^2sat	7
3	$a^{\pm 1}$, $sa^{\pm 1}$, $a^{\pm 1}s$, asa, sas	8
6	$a^{\pm 1}t$, $sa^{\pm 1}t$, $a^{\pm 1}st$, $asat$, $sast$	8
Total number of elements		24

5. $C_2 \times D_6 = \langle a, s, t | a^6 = s^2 = t^2 = (sa)^2 = [a, t] = [s, t] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	sa^i, sta^i, $(i = 0, \ldots, 5)$ a^3, t, ta^3	15
3	$a^{\pm 2}$	2
6	$a^{\pm 1}$, $ta^{\pm 2}$, $ta^{\pm 1}$	6
Total number of elements		24

6. $C_3 \times D_4 = \langle \bar{a}, s | \bar{a}^{12} = s^2 = s\bar{a}s\bar{a}^5 = 1 \rangle$

Elements

Order	Elements	Number of elements
2	s, $\bar{a}^6$, $s\bar{a}^6$, $t\bar{a}^{\pm 3}$	5
3	$\bar{a}^{\pm 4}$	2
4	$\bar{a}^{\pm 3}$	2
6	$\bar{a}^{\pm 2}$, $s\bar{a}^{\pm 1}$, $s\bar{a}^{\pm 2}$, $s\bar{a}^{\pm 4}$, $s\bar{a}^{\pm 5}$	10
12	$\bar{a}^{\pm 1}$, $\bar{a}^{\pm 5}$	4
Total number of elements		24

7. $C_3 \times Q = \langle a, s, t | a^3 = s^4 = t^4 = (st)^4 = [a, s] = [a, t] = 1, s^2 = t^2 \rangle$

Elements

Order	Elements	Number of elements
2	s^2	1
3	$a^{\pm 1}$	2
4	$s^{\pm 1}$, $t^{\pm 1}$, $s^{\pm 1}t$	6
6	$a^{\pm 1}s^2$	2
12	$a^{\pm 1}s^{\pm 1}$, $a^{\pm 1}t^{\pm 1}$ $a^{\pm 1}s^{\pm 1}t$	12
Total number of elements		24

8. $C_4 \times D_3 = \langle a, b, s | a^3 = b^4 = s^2 = (sa)^2 = [a, b] = [b, s] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	$sa^{\pm 1}$, b^2, sb^2, $sa^{\pm 1}b^2$	7
3	$a^{\pm 1}$	2
4	$b^{\pm 1}$, $sb^{\pm 1}$, $sa^{\pm 1}b^{\pm 1}$	8
6	$a^{\pm 1}b^2$	2
12	$a^{\pm 1}b^{\pm 1}$	4
Total number of elements		24

9. $C_6 \rtimes C_4 = \langle a, b | a^6 = b^4 = 1, b^3ab = a^5 \rangle$

Elements

Order	Elements	Number of elements
2	a^3, b^2, b^2a^3	3
3	$a^{\pm 2}$	2
4	$b^{\pm 1}a^i$, $(i = 0, \ldots, 5)$	12
6	$s^{\pm 1}$, $t^2s^{\pm 1}$, $t^2s^{\pm 2}$	6
Total number of elements		24

10. $D_{12} = \langle a, s | a^{12} = s^2 = (sa)^2 \rangle$

11. $\Sigma_4 = \langle s, t | s^4 = t^2 = (ts)^3 \rangle$

Elements

Order	Elements	Number of elements
2	$t, s^2, s^2ts^2, s^3ts, ts^2t$ $ts^2ts, ts^2ts^2, ts^2ts^3$	9
3	$ts, st, ts^3, s^2ts, s^3ts^2$ sts^2, s^2ts^3, s^3t	8
4	$ts^2, s^2t, s^{\pm 1}, s^3ts^3, sts$	6
Total number of elements		24

12. $\langle 2,3,3\rangle = \langle a,s,t|a^3 = t^4 = s^4 = (st)^4 = 1, s^2 = t^2, a^2sa = t, a^2ta = st\rangle = Q \rtimes C_3$

Elements

Order	Elements	Number of elements
2	s^2	1
3	$a^{\pm 1}, s^3a^{\pm 1}, sta^m \pm 1, s^2ta^{\pm 1}$	8
4	$t^{\pm 1}, s^{\pm 1}, s^{\pm 1}t$	6
6	$ta^{\pm 1}, sa^{\pm 1}, s^2a^{\pm 1}, s^3ta^{\pm 1}$	8
Total number of elements		24

13. $\langle 4,6|2,2\rangle = \langle s,t|s^4 = t^6 = (st)^2 = (s^{-1}t)^2\rangle$

Elements

Order	Elements	Number of elements
2	$s^2, t^3, s^2t^3, s^{\pm 1}t^{\pm 1}, s^{\pm 1}t^3$	9
3	$t^{\pm 2}$	2
4	$s^{\pm 1}, s^{\pm 1}t^{\pm 2}$	6
6	$t^{\pm 1}, s^2t^{\pm 1}, s^2t^{\pm 2}$	6
Total number of elements		24

14. $\langle -|2,2,3\rangle = \langle a,t|a^4 = t^4 = (at)^6, a^2 = t^2\rangle$

Elements

Order	Elements	Number of elements
2	$a^2, tatata, tatata^3$	3
3	$tata, atat$	2
4	$a^{\pm 1}, t^{\pm 1}, tat, tatat, ata, atata$ $tata^2, tatata^2, ata^3, atata^3$	12
6	$at, ta, ta^3, ata^2, tata^3, atata^2$	6
Total number of elements		24

15. $\langle 2,2,6\rangle = \langle a,t|a^{12} = t^4 = 1, t^2 = a^6, t^3at = a^5\rangle$

Elements

Order	Elements	Number of elements
2	$t^2, t^{\pm 1}a, t^{\pm 1}a^3, t^{\pm 1}a^5$	7
3	$a^{\pm 4}$	2
4	$a^{\pm 3}, t^{\pm 1}, t^{\pm 1}a^2, t^{\pm 1}a^4$	8
6	$a^{\pm 2}$	2
12	$a^{\pm 1}, a^{\pm 5}$	4
Total number of elements		24

Groups of order 27

1. $C_{27} = \langle a | a^{27} = 1 \rangle$
2. $C_9 \times C_3 = \langle a, b | a^9 = b^3 = [a, b] = 1 \rangle$
3. $C_3 \times C_3 \times C_3 = \langle a, b, c | a^3 = b^3 = c^3 = [a, b] = [b, c] = [a, c] = 1 \rangle$
4. $C_9 \rtimes C_3 = \langle a, b | a^9 = b^3 = 1, b^2 ab = a^7 \rangle$

Elements

Order	Elements	Number of elements
3	$b^{\pm 1}, a^{\pm 3}, b^{\pm 1} a^{\pm 3}$	8
9	$a^{\pm 1}, a^{\pm 2}, a^{\pm 4}$ $b^{\pm 1} a^{\pm 1}, b^{\pm 1} a^{\pm 2}, b^{\pm 1} a^{\pm 4}$	18
Total number of elements		27

5. $(C_3 \times C_3) \rtimes C_3 = \langle a, b, c | a^3 = b^3 = c^3 = [a, b] = [a, c] = 1, c^2 bc = ab \rangle$

Elements

Order	Elements	Number of elements
3	$a^i b^j c^k$	26
Total number of elements		27

Groups of order 28

1. $C_{28} = \langle a | a^{28} = 1 \rangle$
2. $C_{14} \times C_2 = \langle a, s | a^{14} = s^2 = [a, s] = 1 \rangle$
3. $D_{14} = \langle a, s | a^{14} = s^2 = (sa)^2 = 1 \rangle$
4. $\langle 2, 2, 7 \rangle = \langle a, b | a^{14} = b^4 = 1, a^7 = b^2, b^3 ab = a^{13} \rangle$

Elements

Order	Elements	Number of elements
2	$b^2 = a^7$	1
4	ba^i, $(i = 0, \ldots, 13)$	14
7	$a^{\pm 2}, a^{\pm 4}, a^{\pm 6}$	6
14	$a^{\pm 1}, a^{\pm 3}, a^{\pm 5}$	6
Total number of elements		28

Groups of order 30

1. $C_{30} = \langle a | a^{30} = 1 \rangle$
2. $D_{15} = \langle a, s | a^{15} = s^2 = (sa)^2 = 1 \rangle$
3. $D_5 \times C_3 = \langle a, b, s | a^5 = s^2 = (sa)^2 = b^3 = [a, b] = [b, s] = 1 \rangle$

4. $D_3 \times C_5 = \langle a, b, s | a^3 = s^2 = (sa)^2 = b^5 = [a, b] = [b, s] = 1 \rangle$

Groups of order 36

1. $C_{36} = \langle a | a^{36} = 1 \rangle$
2. $C_{18} \times C_2 = \langle a, s | s^2 = a^{18} = [s, a] = 1 \rangle$
3. $C_{12} \times C_3 = \langle a, b | b^3 = a^{12} = [a, b] = 1 \rangle$
4. $C_6 \times C_6 = \langle a, b | a^6 = b^6 = [a, b] = 1 \rangle$
5. $(C_2 \times C_2) \rtimes C_9 = \langle a, s, t | a^9 = s^2 = t^2 = [s, t] = 1, a^8 sa = t, a^8 ta = st \rangle$

Elements

Order	Elements	Number of elements
2	s, t, st	3
3	$a^{\pm 3}$	2
6	$s^i t^j a^{\pm 3}$, $(i, j) \neq (0, 0)$	6
9	$s^i t^j a^{\pm 1}, s^i t^j a^{\pm 2}, s^i t^j a^{\pm 4}$	24
Total number of elements		36

6. $A_4 \times C_3 = \langle a, b, s | a^3 = b^3 = s^2 = [a, b] = [s, b] = (as)^3 = 1 \rangle$

Elements

Order	Elements	Number of elements
2	s, asa^2 a^2sa	3
3	$b^{\pm 1}$, $a^{\pm 1} b^i$, $sa^{\pm 1} b^i$ $a^{\pm 1} sb^i$, $(sas)^{\pm 1} b^i$	26
6	$sb^{\pm 1}$, $asa^2 b^{\pm 1}$ $a^2 sab^{\pm 1}$	6
Total number of elements		36

7. $C_9 \rtimes C_4 = \langle a, t | a^9 = t^4 = 1, t^3 at = a^8 \rangle$

Elements

Order	Elements	Number of elements
2	t^2	1
3	$a^{\pm 3}$	2
4	$a^i t^{\pm 1}$, $(i = 0, \dots, 8)$	18
6	$a^{\pm 3} t^2$	2
9	$a^{\pm 1}, a^{\pm 2}, a^{\pm 4}$	4
18	$a^{\pm 1} t^2, a^{\pm 2} t^2, a^{\pm 4} t^2$	6
Total number of elements		36

8. $D_{18} = \langle a, s | a^{18} = s^2 = (sa)^2 \rangle$

9. $(C_3 \times C_3) \rtimes_1 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = [a, b] = 1, t^3 a t = a^2, t^3 b t = b^2 \rangle$

Elements

Order	Elements	Number of elements
2	t^2	1
3	$a^i b^j$, $(i, j) \neq (0, 0)$	8
4	$t^{\pm 1}$	2
6	$a^i b^j t^2$, $(i, j) \neq (0, 0)$	8
18	$a^i b^j t^{\pm 1}$, $(i, j) \neq (0, 0)$	16
Total number of elements		36

10. $(C_3 \times C_3) \rtimes_3 C_4 = \langle a, b, t | a^3 = b^4 = t^3 = [a, t] = [b, t] = 1, b^3 a b = a^2 \rangle = T \times C_3$

Elements

Order	Elements	Number of elements
2	b^2	1
3	$a^i t^j$, $(i, j) \neq (0, 0)$	8
4	$a^i b^{\pm 1}$	6
6	$a^{\pm 1} b^2, t^{\pm 1} b^2$	4
18	$a^i b^{\pm 1} t^j$, $(i, j) \neq (0, 0)$	16
Total number of elements		36

11. $(C_3 \times C_3) \rtimes_2 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = [a, b] = 1, t^3 a t = b, t^3 b t = a^2 \rangle$

Elements

Order	Elements	Number of elements
2	$a^i b^j t^2$	10
3	$a^i b^j$, $(i, j) \neq (0, 0)$	8
4	$a^i b^j t^{\pm 1}$	18
Total number of elements		36

12. $\langle 3, 3, 3, 2 \rangle \times C_2 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 = [a, b] = [a, t] = [b, t] = [s, t] = 1, sas = a^2, sbs = b^2 \rangle$

Elements

Order	Elements	Number of elements
2	$a^i b^j s, a^i b^j st, t$	19
3	$a^i b^j$, $(i, j) \neq (0, 0)$	8
4	$a^i b^j t$	8
Total number of elements		36

13. $C_6 \times D_3 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^3 = (sa)^2 = [a, b] = [a, t] = [b, s] = [b, t], [s, t] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	t, sa^i, sta^i	7
3	a^ib^j, $(i, j) \neq (0, 0)$	8
6	ta^ib^j, $(i, j) \neq (0, 0)$ $sb^{\pm1}, sa^{\pm1}b^{\pm1}, sta^{\pm1}b^{\pm1}$	20
Total number of elements		36

14. $D_3 \times D_3 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 = (st)^2 = [a, b] = [a, t] = [b, s] = (sa)^2 = (tb)^2 \rangle$

Elements

Order	Elements	Number of elements
2	a^is, b^it, a^ib^jst	15
3	a^ib^j	8
6	$a^ib^{\pm1}s, a^{\pm1}b^jt$	12
Total number of elements		36

Groups of order 45

1. $C_{45} = \langle a | a^{45} \rangle$
2. $(C_3 \times C_3) \rtimes C_5 = \langle a, b, t | a^3 = b^3 = t^5 = [a, b] = 1, t^4at = a^2, t^4bt = b^2 \rangle$

Groups of order 60

1. $C_{60} = \langle u | u^{60} = 1 \rangle$
2. $A_5 = \langle a, b | a^2 = b^3 = (ab)^5 \rangle$
   or sometimes $A_5 = \langle a, b, c | a^2 = b^3 = (c)^5 = abc = 1 \rangle$
3. $C_{30} \times C_2 = \langle a, s | s^2 = a^{30} = [a, s] = 1 \rangle$
4. $C_5 \times A_4 = \langle a, b, s, | a^3 = s^2 = (as)^3 = b^5 = [a, u] = [s, u] = 1 \rangle$
5. $C_{15} \rtimes_{14} C_4 = \langle a, t | a^{15} = t^4 = 1, t^3at = a^{14} \rangle$
6. $C_{15} \rtimes_2 C_4 = \langle a, t | a^{15} = t^4 = 1, t^3at = a^2 \rangle$
7. $C_{15} \rtimes_4 C_4 = \langle a, t | a^{15} = t^4 = 1, t^3at = a^4 \rangle$
8. $C_{15} \rtimes_{11} C_4 = \langle a, t | a^{15} = t^4 = 1, t^3at = a^{11} \rangle$
9. $C_{15} \rtimes_{13} C_4 = \langle a, t | a^{15} = t^4 = 1, t^3at = a^{13} \rangle$
10. $D_{30} = \langle a, s | a^{15} = s^2 = (as)^2 \rangle$

11. $D_3 \times D_5 = \langle a, b, s, t | a^3 = b^5 = s^2 = t^2 = (as)^2 = (bt)^2 = [a, b] = [b, s] = [a, t] = [s, t] = 1\rangle$

12. $D_3 \times C_{10} = \langle a, b, s | a^3 = s^2 = b^{10} = (as)^2 = [a, b] = [s, b] = 1\rangle$

13. $C_6 \times D_5 = \langle a, b, s | a^5 = s^2 = b^6 = (as)^2 = [a, b] = [s, b] = 1\rangle$

Groups of order 72

There are 50 groups of order 72. However, we will only present the groups that are relevant in this case.

1. $\Sigma_4 \times C_3 = \langle t, a, b | t^2 = a^3 = b^3 = (ta)^4 = [t, b] = [a, b] = 1\rangle$

2. $(C_3 \times C_3) \rtimes_8 C_8 = \langle a, b, t | a^3 = b^3 = [a, b] = t^8 = 1, t^7 at = b, t^7 bt = ab\rangle$

3. $\big((C_3 \times C_3) \rtimes_1 C_4\big) \times C_2 = \langle a, b, s, t | a^3 = b^3 = [a, b] = t^4 = s^2 = [a, s] = [b, s] = [t, s] = 1, t^3 at = a^2, t^3 bt = b^2\rangle$

4. $\big((C_3 \times C_3) \rtimes_2 C_4\big) \times C_2 = \langle a, b, s, t | a^3 = b^3 = [a, b] = t^4 = s^2 = [a, s] = [b, s] = [t, s] = 1, t^3 at = b, t^3 bt = a^2\rangle$

5. $(C_3 \times C_3) \rtimes_1 D_4 = \langle a, b, s, t | a^3 = b^3 = [a, b] = t^4 = s^2 = (st)^2 = [s, a] = [s, b] = 1, t^3 at = b, t^3 bt = a\rangle = T \rtimes C_6$

6. $(C_3 \times C_3) \rtimes_2 D_4 = \langle a, b, s, t | a^3 = b^3 = [a, b] = t^4 = s^2 = (st)^2 = (sa)^2 = [s, b] = 1, t^3 at = b, t^3 bt = a^2\rangle$

7. $(C_3 \times C_3) \rtimes_1 Q = \langle a, b, s, t | a^3 = b^3 = s^4 = t^4 = [a, b] = [b, t] = [a, s] = (ts)^2 = 1, t^2 = s^2, t^{-1} at = a^{-1}, s^{-1} bs = b^{-1}\rangle$

8. $(C_3 \times C_3) \rtimes_2 Q = \langle a, b, s, t | a^3 = b^3 = s^4 = t^4 = [a, b] = [a, s] = [b, t] = 1, t^2 = s^2, s^{-1} ts = t^{-1}, s^{-1} bs = b^{-1}, t^{-1} at = a^{-1}\rangle$

9. $(C_3 \times C_3) \rtimes_3 Q = \langle a, b, s, t | a^3 = b^3 = s^4 = t^4 = [a, b] = [a, b] = [b, t] = [a, s] = 1, t^2 = s^2, s^{-1} ts = t^{-1}, s^{-1} bs = b^{-1}\rangle$

10. $(C_3 \times C_3) \rtimes_4 Q = \langle a, b, s, t | a^3 = b^3 = s^4 = t^4 = [a, b] = [a, t] = [b, t] = 1, t^2 = s^2, s^{-1} bs = b^{-1}, s^{-1} as = a^{-1}\rangle$

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