# ALGEBRAIC GEOMETRY AND RIEMANN SURFACES

DANIEL YING

ABSTRACT. In this thesis we will give a present survey of the various methods used in dealing with Riemann surfaces. Riemann surfaces are central in mathematics because of the multiple connections between complex analysis, algebraic geometry, hyperbolic geometry, group theory, topology etc.

The main focus is the connection of holomorphic morphisms with branched coverings, and the use of permutation groups in classifying these morphisms.

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## 1. **Introduction**

Why are Riemann surfaces important?

Riemann surfaces originated in complex analysis when working with multi-valued functions. The multi-valued functions arise because of the analytic continuation that take place in the complex plane. Normally, one could work around this problem by defining the function to have just one of the values. For example, the expression $y = \sqrt{x}$, when studied as a function, we simply choose $f(x) = \sqrt{x}$ for real $x$. That is we select one branch of the function (conventionally the positive square root). Now this is a continuous function from $[0, \infty] \rightarrow \mathbb{R}$, which is analytical everywhere except at 0.

In the complex case, working with $w = \sqrt{z}$, we can simply take the expression $w^2 = z$, and then make some choice in order to get a single valued function. For this choice to also be continuous we also have to make a cut in the domain.

One way of doing this is to define the function $f : \mathbb{C}\backslash\mathbb{R} \rightarrow \mathbb{C}$ to be the square root with positive real part. This is uniquely defined, away from the negative real axis, and is continuous and in fact complex analytic away from the negative real axis.

Another way, is to let $f$ be the square root with positive imaginary part. This function is again uniquely defined, away from the positive real axis this time, and determines a complex analytic function away from the positive real axis.

Put in a more mathematical way, if $z = re^{i\phi}$ we define the two branches

(1) $\sqrt{z} = \sqrt{r}e^{i\phi/2}$ for $-\pi < \phi < \pi$ and
(2) $\sqrt{z} = \sqrt{r}e^{i\phi/2}$ for $0 < \phi < 2\pi$.

In either cases there is no way of extending the function continuously to the missing half line in question, because as the value of $z$ approaches the line from two different directions, the value of $\sqrt{z}$ differs by a sign.

The idea of Riemann was to replace the domain of the function in question with a many sheeted covering of the complex plane. If the covering is constructed such that over any given point in the plane there are as many points in the covering as function elements at that point, then on the covering surface the analytic function becomes single valued.

The algebraic geometry comes in when we start to investigate the covering surfaces. The fundamental groups of the original surface will have properties, that will be contained in the fundamental group of the covering surface and we will work out how the two groups are connected. One of the main theorems here, is Poincarè's Theorem, which states that given a polygon with some side pairing maps, we can generate a group $G$ such that when $G$ is applied to the polygon, it will tessellate some domain.

We will start with topological surfaces and the use of the Euler characteristic. Then we will develop the theory for coverings and branched coverings which are helpful in order to understand Riemann surfaces. We will show that for any Riemann surface there is a unique algebraic group (up to isomorphism) corresponding to the surface. Such groups are always Fuchsian groups. Therefore the theory for such groups are needed when describing Poincaré's theorem.

The last chapter is about how to get from a Riemann surface to an algebraic curve. The main theorem for this is the famous Riemann normalization theorem. This is not as intuitive as going from an algebraic curve to a Riemann surface. As it turns out the way to get there is very complex.

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## 2. **Surfaces**

### Topological Surfaces and the Euler Characteristic

The surfaces that are of our interest are surfaces that are connected Hausdorff spaces which satisfy the second axiom of countability and has an analytical structure. Therefore, all surfaces mentioned here are assumed to be of this type.

For a smooth compact surface $X$ of genus $g$, (such a surface is commonly denoted by $X_g$), we can find a triangulation such that, the number of edges, vertices and faces ($e$,$v$ and $f$) satisfy[1]

$$v - e + f = 2 - 2g \tag{2.1}$$

The number $2 - 2g$ is independent of the triangulation of the surface and is called the Euler Characteristic of the surface, $\chi(X)$.

We can cover surfaces with other surfaces described in the following definition. This will be discussed more in later chapter about smooth coverings and branched coverings.

**Definition 2.1.** If $X$ and $Y$ are topological spaces (surfaces), a covering map is a continuous mapping $p : Y \to X$ with the property that each point of $X$ has an open neighborhood $N$ such that $p^{-1}(N)$ is a disjoint union of open sets, each of which is homeomorphically mapped by $p$ onto $N$.

We say that the cardinality, $n$, of the set $\{p^{-1}(x)\}$ is the number of sheets of the covering. That is $n = |\{p^{-1}(x)\}|$.

### Riemann surfaces

There are some surfaces which we are interested in that were studied by Riemann, which surprisingly have been named Riemann surfaces. We begin with a definition of such surfaces.

**Definition 2.2.** A Hausdorff connected topological space $X$ is a Riemann surface if there exists a family

$$\{(\phi_j, U_j) : j \in J\}$$

called an atlas (each $(\phi_j, U_j)$ is called a chart) such that

(1) $\{U_j : j \in J\}$ is an open cover of $X$.
(2) each $\phi_j$ is a homeomorphism of $U_j$ onto an open subset of the complex plane.
(3) if $U \subseteq U_i \bigcap U_j \neq \varnothing$, then

$$\phi_i \phi_j^{-1} : \phi_j(U) \to \phi_i(U)$$

is a holomorphic[2] map between the plane sets $\phi_j(U)$ and $\phi_i(U)$.

(1)states that X is covered by a collection of open sets each of which by (2) is homeomorphic to an open subset of $\mathbb{C}$. The two distinguished sets may overlap but then by (3) the corresponding homeomorphisms are related by an holomorphic homeomorphism.

*Remark* 2.3. Throughout this paper we will use local coordinates on the surfaces, that is we are interested in the local appearance of the surface.

**Example 2.4.** The most trivial example of a Riemann surface is of course the complex plane it self with the identity coordinate chart $\mathbb{C} \to \mathbb{C}$. To see this let $\{U_j : j \in J\}$ be an open cover of the complex plane and let $\phi_j$ send each $U_j$ to itself. From this the result should be obvious. Also any open subset of $\mathbb{C}$ or Riemann surface is also a Riemann surface.

[1] For a proof of this see Fulton [6] p. 113-114
[2] Holomorphic means complex analytic

**Example 2.5.** The sphere $S^2$ is a compact Riemann surface with the charts given by spherical projection from the north and south poles. The below picture shows how it is done.

FIGURE 1. The sphere as a Riemann surface.

By defining two subsets of $S^2$ as the following:

$$U_1 = S^2 \backslash \{S\}$$
$$U_2 = S^2 \backslash \{N\}$$

where $S$ and $N$ denotes the south and north pole respectively. Then the two projection maps $\phi_i : U_i \to \mathbb{C}$, $i = 1, 2$, gives us

$$\phi_1(U_1 \bigcap U_2) = \mathbb{C} \backslash \{0\} = \phi_2(U_1 \bigcap U_2)$$

and the transition of $\phi_1$ and $\phi_2$,

$$\phi_1 \phi_2^{-1} : \mathbb{C} \backslash \{0\} \to \mathbb{C} \backslash \{0\}$$

is given by $z \mapsto 1/z$ which is holomorphic outside 0.

**Example 2.6.** The torus $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}i)$ is a compact Riemann surface, using the projection from small open subsets in $\mathbb{C}$ to the quotient space as coordinate charts.

Example 2.5 is worth examining a little closer. Since the sphere is a compact surface we know that there is a finite open subcover of it. Hence a question that arises is whether the atlas can be taken with any open cover of the surface. This is actually true by (3) in the definition. We will call the set $\{(\phi_j, U_j) : j \in J\}$ a *maximal atlas* of $X$ if it contains all admissible charts of $X$.

*Remark* 2.7. There is another alternative definition of a Riemann surface which states that a Riemann surface $X$ is a connected surface with a special collection of coordinate charts $\varphi_\alpha : U_\alpha \to X$ where $U_\alpha$ is a subset of $\mathbb{C}$, $\{\varphi_\alpha(U_\alpha) : \alpha \in A\}$ is an open cover of $X$, and any change of coordinate mapping from $U_\alpha$ to $U_\beta$ is $\mathcal{C}^\infty$ and analytic.

The coordinate charts $(\varphi_\alpha, U_\alpha)$ in this definition are the atlas $\{(\varphi_\alpha, U_\alpha) : \alpha \in A\}$ for the surface. Simply what the definition states is that for any Riemann surface there is a chart from some subset of the complex plane that maps to some subset of the Riemann surface, and if there is a change of coordinates in the complex plane then this mapping is infinitely differentiable and analytic.

This together with definition 2.2 gives us the nice property of Riemann surfaces, i.e. that locally they are equivalent to the complex plane $\mathbb{C}$. This property gives gives the use of Riemann surfaces when studying multivalued complex functions.

The Riemann surfaces we will study later will sometimes have singular points, which geometrically wont be equivalent to the complex plane (locally at the singular points). For example, if $G$ is a cyclic group acting on the sphere $S^2$ fixing points $x$ and $y$ then the quotient is a surface with the same curvature as $S^2$ at all points except in $x$ and $y$. One can see such a point as having the angle $2\pi/n$, since the length of a circle of radius $r$ in $S^2/G$ centered on, say, $x$ is $1/n$ times the length of a circle of radius r in $S^2$.

Figure 2. Singular points on a surface.

So the Euler characteristic (as we will see on page 14) will in general case become

$$\chi(X) = 2 - 2g - \sum_{i=1}^{s}(1 - \frac{1}{m_i}) \tag{2.2}$$

where $m_i$ is the order of the rotation around the singular point.

We have, in the above example, just touched the main part of the theory of Riemann surfaces, namely the functions between such surfaces. This is a topic that will be examined very closely in the rest of the thesis. However, for now it is just worth noting that the function above (being the quotient map from the surface to the quotient between the surface and the group), seem to wrap the sphere around the surface obtained by the quotient map.

We will start by examining functions on Riemann surfaces.

**Holomorphic and meromorphic functions**

Since Riemann surfaces are locally just an open set in $\mathbb{C}$ many concepts in the study of complex function theory may be defined on these surfaces using local coordinates as long as the object of study is invariant under coordinate changes.

**Definition 2.8.** Suppose $C$ is a Riemann surface and $\{(U_i, z_i)\}$ is its holomorphic coordinate covering. A meromorphic function (holomorphic function) $f$ on $C$ is defined by a family of mappings $f_i : U_i \to \Sigma = \mathbb{C} \bigcup \{\infty\}$ satisfying the following conditions:

(1) $f_i = f_j$ on $U_i \bigcap U_j$ for $U_i \bigcap U_j \neq \varnothing$.
(2) $\forall i$, $f_i \circ z_i^{-1}$ are all meromorphic (resp. holomorphic) functions.

The meromorphic functions on $C$ form a function field denoted $K(C)$ while the holomorphic functions form an algebra over $\mathbb{C}$ denoted by $\mathcal{O}(C)$.

A fact that follows easily by looking at the alternative definition in remark 2.7 is:

If $X$ is a Riemann surface, then any function $f : X \to \mathbb{C}$ is meromorphic. To see this, note that for each coordinate chart $\varphi_\alpha : U_\alpha \to X$, the composition $f \circ \varphi_\alpha$ (a change of coordinate chart) has to be holomorphic. Hence $f$ is a holomorphic function.

By using complex function theory we can show that the only holomorphic functions on a compact Riemann surface $C$ are the constant functions.

**Definition 2.9.** Suppose $C$ is a compact Riemann surface, $f$ a meromorphic function, $p \in C$ and $f \neq 0$. By selecting a local coordinate $z$ in a neighborhood of the point $p$ such that $z(p) = 0$ we get that

$$f = z^v h(z)$$

where $h(z)$ is a holomorphic function, $h(0) \neq 0$, and $v \in \mathbb{Z}$. This $v$ is called the order or multiplicity of $f$ at the point $p$, denoted by $v_p(f)$. When $v_p(f) > 0$, $p$ is called a zero of $f$ and $|v_p(f)|$ is called the order or multiplicity of the zero p; when $v_p(f) < 0$, $p$ is called a pole of $f$ and $|v_p(f)|$ is called the order or multiplicity of the pole $p$.

**Definition 2.10.** Suppose $C$ and $C'$ are Riemann surfaces with $\{(U_i, z_i)\}$ and $\{(U'_a, z'_a)\}$ as their holomorphic coordinate coverings. Then a holomorphic mapping $f : C \to C'$ is by definition a family of continuous mappings

$$f_i : U_i \to C' \ , \forall i$$

such that

(1) $f_i = f_j$ on $U_i \bigcap U_j$ for $U_i \bigcap U_j \neq \varnothing$.
(2) $z'_a \circ f_i \circ z_i^{-1}$ is a holomorphic function on $f^{-1}(U'_a) \bigcap U_i$ whenever $f^{-1}(U'_a) \bigcap U_i \neq \varnothing$

*Remark* 2.11. As we shall see in the next chapter, the holomorphic mapping $f : C \to C'$ is a branched covering of Riemann surfaces and this is why the study of covering maps is important when working with Riemann surfaces.

Now if $X$ and $Y$ are two Riemann surfaces then any mapping between them, say $f : X \to Y$, is holomorphic at a point $P \in X$ if there are charts $\varphi : U \to X$ and $\psi : V \to Y$ mapping to neighborhoods of $P$ and $f(P)$, respectively, so that $f(\varphi(U)) \subset \psi(V)$ and the composite $\psi^{-1} \circ f \circ \varphi$ is a holomorphic function from $U$ to $V$.

This condition is independent of the choice of local coordinate charts $\varphi$ and $\psi$. Thus we can without loss of generality choose $U$ and $V$ to be open discs around the origin. Hence we can write the composition, $h = \psi^{-1} \circ f \circ \varphi$, as a convergent power series

$$h(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Now the smallest integer $e$ such that $a_n = 0$, for all $n \leq e$, is called the *ramification index* of $f$ at $P$ and it is independent of the choice of coordinates [3]. The ramification index is denoted $e_f(P)$ or just $e(P)$ when there is no confusion of which function is being used. The point $P$ is called a ramification point for $f$ if $e_f(P) > 1$.

By looking at definition 2.9 we see that if, say, the ramification index is $e(P) = k$ we get that

[3]See Fulton [6] p. 265-266

$$h(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=k+1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_{n+k} z^{n+k} = z^{k+1} \sum_{n=0}^{\infty} a_{n+k+1} z^n = z^{k+1} g(z).$$

where $g(0) \neq 0$. So the relation between the ramification index, $k$, and the order of multiplicity of $f$ at the point $P$, $v_f(p)$, is given by the following: $v_f(p) = k+1$.

An important consequence of the above discussion is the following lemma.

**Lemma 2.12.** *By a change of coordinates we can write the composition $h = \psi^{-1} \circ f \circ \varphi$ in a local representation as*

$$\tilde{h} = z^e.$$

*The number $e-1$ is the ramification index at the origin.*

*Proof.* From the power series $h(z) = \sum_{n=1}^{\infty} a_n z^n$ we see that if $e \geq 1$ we can write it as

$$h(z) = \sum_{n=0}^{\infty} a_{n+e} z^{n+e} = z^e \sum_{n=0}^{\infty} a_{n+e} z^n = z^e g(z).$$

Where $g(z)$ is a holomorphic function in a neighborhood of the origin and at the origin we have

$$g(0) = \sum_{n=0}^{\infty} a_{n+e} 0^n = a_e \neq 0.$$

Such a holomorphic function can be written as the power of a holomorphic function $k(z)$. Thus we have that

$$h(z) = (z \cdot k(z))^e$$

for some holomorphic function $k$ with $k(0) \neq 0$. (Note that this just shows what we have stated in definition 2.9)

Now the mapping $z \mapsto z \cdot k(z)$ is a holomorphic isomorphism, since the derivative does not vanish at the origin. Hence we can define $\widetilde{\varphi}$ such that

$$\widetilde{\varphi}(z) = \varphi(z \cdot k(z))$$

for all sufficiently small $z$, and so we get that $\tilde{h} = \psi^{-1} \circ f \circ \widetilde{\varphi}$ maps $z \mapsto z^e$.

Now $z \mapsto z^e$ maps $0 \mapsto 0$ and outside the origin it is an $e$-sheeted covering. That is, there are neighborhoods $U$ of $P$ and $V$ of $f(P)$ such that $f(U) = V$ and $U \backslash \{P\} \mapsto V \backslash \{f(P)\}$ is an $e$-sheeted covering. Thus the number $e$ is independent of the choice of coordinates. In fact $e$ depends only on the topology of the map $f$ near $P$.

□

We now state an important result, which is basic to the theory, due to F. Klein, H. Poincaré and P. Kobe, called the *Uniformisation Theorem*:

**Theorem 2.13.** *Every simply connected Riemann surface is conformally equivalent to just one of:*

(1) *the Riemann sphere;*
(2) *the complex plane;*
(3) *the open unit disc.*

*Remark* 2.14. This is a consequence of lemma 2.12

**Theorem 2.15.** *Let $X$ be a compact Riemann surface. If $f \in K(X)$ is not a constant function, then*

$$\sum_{p \in X} v_p(f) = 0.$$

*Proof.* We will use the differential $\omega = df/f$ and from the residue theorem we get that

$$\sum_{p \in X} Res_p(\omega) = \sum_{p \in X} \frac{1}{2\pi i} \oint_{\gamma_p} \omega = 0$$

where $\gamma_p$ is a small circle around the poles $p$ in $X$.

Now, by using the fact that $Res_p(\omega) = Res_p(\frac{df}{f(z)})$ =(the order of $f$ at $p$) $= v_p(f)$, we get that

$$\sum_{p \in X} Res_p(\omega) = \sum_{p \in X} v_p(f) = 0$$

□

**Corollary 2.16.** *With the same notation as in the previous theorem, the number of zeros of $f$ is equal to the number of poles of $f$.*

**Corollary 2.17.** *If $f \in K(X)$ is not constant, then for any $a \in X$, we have*

$$|f^{-1}(a)| = |\{poles\ of\ f\}|$$

**Definition 2.18.** An *isomorphism* (a *biholomorphic* map) between Riemann surfaces is a holomorphic map $F : X \to Y$ which is bijective, and whose inverse is also holomorphic. A self-isomorphism $F : X \to X$ is called an *automorphism* of $X$.

The set of automorphisms of a surface, $X$, form a group, under composition, called the group of automorphisms of $X$, denoted $Aut(X)$.

*Remark* 2.19. As we shall see in chapter 3, automorphisms of a Riemann surface corresponds to regular branched coverings, associated to the action of the group of automorphisms.

## 3. **Smooth and Branched Coverings**

### Smooth Coverings

Coverings arise naturally when working with Riemann surfaces, since the atlases used are in fact coverings of the Riemann surface. However there are many more such coverings so before we give the proper definition we will start of with an easy example of a covering.

**Example 3.1.** Define a map $f : H \to C_u$ from the helix $H = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) = (cos(t), sin(t), t), t \in \mathbb{R}\}$ to the unit circle $C_u$ by

$$f(x, y, z) = (x, y, 0)$$

FIGURE 3. The map $f$ from the helix to the unit circle.

This is in fact a covering of the unit circle so the helix is called a covering space of the unit circle.

**Definition 3.2.** A continuous map $f : Y \to X$ between two surfaces is called a (smooth) covering if for every $x \in X$ there is a connected open neighborhood $\Delta$ of $x$ such that

$$f^{-1}(\Delta) = \bigcup_{i=1}^{n or \infty} \Delta_i$$

where the $\Delta_i$'s are the components of $f^{-1}(\Delta)$ and $f : \Delta_i \to \Delta$ is a homeomorphism

The number of components of $f^{-1}(\Delta)$ is called the number of sheets of the covering. This is similar to the previous description of a covering, and so if there are $n$ number of components, then we say that the covering is an $n$-sheeted covering.

**Example 3.3.** Take $f : \mathbb{C} \to T$ (the torus) such that $(x, y) \mapsto (\overline{x}, \overline{y})$, where $\overline{x} = x \ mod(\tau_1)$ and $\overline{y} = y \ mod(\tau_2)$ for some lattice generated by $\tau_1$ and $\tau_2$.

FIGURE 4. The torus as a quotient space

This covering is easily seen to be infinitely sheeted.

**Example 3.4.** The antipodal map applied to the Riemann sphere gives us a two sheeted covering of the projective plane $\mathbb{P}^2$.

FIGURE 5. The projective plane

Next we will be interested in the fundamental group of covering spaces. It turns out that the group of a covering space viewed as a quotient space is a subgroup of the group of the surface it covers. But to show this we need to know what happens to paths in the covered surface $X$.

**Definition 3.5.** Suppose $f : Y \to X$ is a local homeomorphism between surfaces. Let $\gamma : [0, 1] \to X$ be a continuous path (map) starting at $\gamma(0) = x_0$. Then the lift of $\gamma$ is defined to be the path $\widetilde{\gamma} : [0, 1] \to Y$ such that $\widetilde{\gamma}(0) = y_0$, $f(y_0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$.

FIGURE 6. Lifts on a covering surface.

Since the cardinality of $f^{-1}(x_0)$ is equal or greater than one there are several lifts corresponding to a path in $X$. We therefore say that $\widetilde{\gamma}$ is the lift of $\gamma$ starting at $y_0$. If $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are two lifts starting at $y_0$ then $\widetilde{\gamma} = \widetilde{\gamma}'$ (the uniqueness of lifts).

**Theorem 3.6.** *(The Monodromy Theorem): Let $f : Y \to X$ be a covering. Let $\gamma_0$, $\gamma_1$ be paths in $X$ starting and ending at the same points so that $\gamma_0$ is homotopic to $\gamma_1$. Let $\widetilde{\gamma_0}$ and $\widetilde{\gamma_1}$ be lifts of $\gamma_0$ and $\gamma_1$ starting at the same point above $\gamma_0(0)$. Then $\widetilde{\gamma_0}$ is homotopic to $\widetilde{\gamma_1}$, and in particular $\widetilde{\gamma_0}(1) = \widetilde{\gamma_1}(1)$*

The theorem will not be proved here but is straight forward from the definition of homotopy[4]. Next we will define a covering in a more useful way and give a classification of smooth coverings.

Let $X$ be a surface, $x_0 \in X$. A triple $(Y, f, y_0)$ will be called a covering of the surface $(X, x_0)$ if $f : Y \to X$ is a covering of surfaces, $y_0 \in Y$ and $f(y_0) = x_0$. Two coverings, $(Y, f, y_0)$ and $(Z, g, z_0)$ are said to be equivalent if there is a homeomorphism, $h : Y \to Z$ between them such that $h(y_0) = z_0$ and the composition $g \circ h = F$.

Suppose $(Y, f, y_0)$ is a covering of $(X, x_0)$. Let $[\gamma]$ be an equivalence class of paths in the fundamental group $\pi_1(X, x_0)$ of the surface $X$. Let $D \subset \pi_1(X, x_0)$ be the paths $[\gamma]$ so that the lifts of $\gamma$ starting at $y_0$ is a closed path. By the Monodromy Theorem $D$ is a well defined subgroup of $\pi_1(X, x_0)$. (In fact if we define $f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ by $f_*([\gamma]) = [f \circ \gamma]$ then $D = f_*(\pi_1(Y, y_0))$).

**Theorem 3.7.** *(Existence Theorem). If $D$ is a subgroup of $\pi_1(X, x_0)$ then there exists a covering $(Y, f, y_0)$ of $(X, x_0)$ such that $D = f_*(\pi_1(Y, y_0))$.*

**Theorem 3.8.** *Suppose $D \subset D' \subset \pi_1(X, x_0)$, so that by the existence theorem, $(Y, f, y_0)$ and $(Y', f', y_0')$ are coverings of $(X, x_0)$. Then there exists a covering $g : Y \to Y'$ such that $(Y, f, y_0)$ is a covering of $(Y', f', y_0')$ and $f = f \circ g$.*

From this theorem we see that there is a partial ordering of the coverings of a surface. That is, we can define that $(Y, f, y_0) < (Y', f', y_0')$ if there is a covering $g : Y \to Y'$ such that $(Y, g, y_0)$ is a covering of $(Y', y_0')$ and $f = f' \circ g$.

This means that if $D \leqslant D' \leqslant \pi_1(X)$ then there is a chain

$$Y \xrightarrow{g} Y' \xrightarrow{f'} X$$

**Corollary 3.9.** *Two coverings $(Y, f, y_0)$ and $(Y', f', y_0')$ are equivalent coverings of $(X, x_0)$ if and only if $D = D'$*

Since a covering is related to its group $D$ we sometimes write a covering as $(Y_D, f_D, y_D)$. A type of coverings that will be interesting in this thesis are the coverings corresponding to $D = \langle e \rangle \subset \pi_1(X, x_0)$. Such a covering is called the *universal covering* of $X$. If the fundamental group of the covering space is elementary, the covering space, $Y_{<e>}$, is simply connected. This can easily be seen by using the uniformisation theorem and the monodromy theorem. We will usually denote the universal covering space by $\mathcal{U}$.

**Example 3.10.** From example 3.1 we see that clearly the helix is simply connected so it is a universal covering of the unit circle.

[4] See Fulton [6]

**Example 3.11.** An example of a covering surface which is not simply connected is the following covering from a surface of genus 3 to a surface of genus 2. The rotation $\Gamma$ has no fixed points, and the axis goes through the center hole of the surface.

FIGURE 7. A covering of the double torus.

**Definition 3.12.** Suppose $f : Y \to X$ is a covering. Then a homeomorphism $g : Y \to Y$ is called a *deck transformation* (or cover transformation) if $g \circ f = f$.

Under composition the set of all deck transformations form a group which is called *the group of deck transformations* and is denoted by $G(Y/X)$. In fact, if $D = f_*(\pi_1(Y, y_0))$ then

$$G(Y/X) \cong N(D)/D.$$

If $D$ is a normal subgroup of the fundamental group of $X$, then clearly we get that $G(Y/X) \cong \pi_1(X, x_0)/D$. As we are interested in the automorphism group of the surface we want to work with those cases where the subgroups are normal. Such coverings are called *regular coverings*.

Another way of seeing the group of deck transformation in regular coverings is to let $G_1$ and $G$ be the fundamental groups of $Y$ and $X$ in the covering $f : Y \to X$. Then the group of deck transformations is

$$G(Y/X) \cong G/G_1.$$

**Example 3.13.** An intuitive way of seeing the deck transformations is given by the following picture (Here $f : Y \to X$ is a covering):

FIGURE 8. Deck transformation.

### The Monodromy Representation of a covering map

By looking at example 3.13 we see that the group of deck transformations $G/G_1$ acting on the fibre $f^{-1}(x_0)$ permutes the fibre (so is a permutation group). Thus, it is natural that we can find an epimorphism

$$\varphi : G \to \Sigma_{|G:G_1|}.$$

To see this, let $f : Y \to X$ be a covering of finite degree $n$. If $f$ corresponds to a subgroup $H \subseteq \pi_1(X, x_0)$ then $n$ is clearly the index of the subgroup $H$.

Consider the fiber $I = f^{-1}(x_0) = \{y_0, ..., y_n\}$ over $x_0$. Every loop $\gamma$ in $X$ based at $x_0$ can be lifted to $n$ paths $\widetilde{\gamma}_0, ..., \widetilde{\gamma}_n$ in $Y$, where $\widetilde{\gamma}_i$ is the lift of $\gamma$ starting at $y_i$.

Next we consider the endpoints $\widetilde{\gamma}_i(1)$, these also lie over $x_0$ and form the entire fibre $I$. Hence $I = \{\widetilde{\gamma}_i(1) : 0 \leq i \leq n\}$ and we denote the point $\widetilde{\gamma}_i(1)$ by $x_{\sigma(i)}$.

The function $\sigma$ is a permutation of the indices $\{0, ..., n\}$ and by the monodromy theorem it is a well defined function depending only on the homotopy class of the loop $\gamma$. Hence we have the epimorphism (or group homomorphism)

$$\varphi : \pi_1(X, x_0) \to \Sigma_n$$

where $\Sigma_n$ denotes the group of symmetries of all permutations on $\{1, ..., n\}$.

*Remark* 3.14. In general $G/G_1$ is not a group, however the deck transformations are given by the epimorphism $\varphi$. With this fact and by using counting methods of algebra we are able to deal with cases with non-normal subgroups.

**Definition 3.15.** The monodromy representation of a covering map $f : Y \to X$ of finite degree $n$ is the epimorphism $\varphi : \pi_1(X, x_0) \to \Sigma_n$ defined above.

Recall from group theory that a subgroup $H \subseteq \Sigma_n$ is said to be *transitive* if for any pair of indices $i$ and $j$ there is a permutation $\sigma$ in the subgroup $H$ which sends $i$ to $j$. So if $Y$ is connected we get the following property of the monodromy representation.

**Lemma 3.16.** *Let $\varphi : \pi_1(X, x_0) \to \Sigma_n$ be the monodromy representation of a covering map $f : Y \to X$ of finite degree $n$, with $Y$ connected. Then the image of $\varphi$ is a transitive subgroup of $\Sigma_n$.*

*Proof.* By using the same notation as above, we consider two points $y_i$ and $y_j$ in the fiber of $f$ over $x_0$. Since $Y$ is connected, there is a path $\widetilde{\gamma}$ in $Y$ starting at $y_i$ and ending at $y_j$. Let $\gamma = f \circ \widetilde{\gamma}$ be the image of $\widetilde{\gamma}$ in $X$. Note that $\gamma$ is a loop in $X$ based at $x_0$ since both $y_i$ and $y_j$ map to $x_0$ under $f$. Hence by construction of $\varphi$, $\varphi([\gamma])$ is a permutation which sends $i$ to $j$. □

**Definition 3.17.** The transitive subgroup of $\Sigma_n$ mentioned above is called the monodromy group of the covering and is denoted by $M(Y/X)$.

### Branched Coverings

Branched coverings are maps which locally are equivalent to $z \mapsto z^n$. That is there are points on the surface with ramification index greater than one.

**Example 3.18.** Consider the map $f : \mathbb{D}\backslash\{0\} \mapsto \mathbb{D}\backslash\{0\}$ from the unit disc to the unit disc (with the origin removed) given by $f(z) = z^n$ a smooth covering of surfaces. If we would use the whole unit disc then clearly the origin would have ramification index $n$.

We can extend $f$ to a holomorphic map $f^e : \mathbb{D} \mapsto \mathbb{D}$ by sending $0 \mapsto 0$ (n times). Then $f^e$ is a branched covering. In terms of the earlier definitions we see that a closed path in the circle sector of angle $2\pi/n$ are mapped into $n$ lifts in the unit disc and a path from one radial side to the other in the circle sector is mapped to a path going once around the origin of the unit disc.

*Remark* 3.19. Note that the covering is from $\mathbb{D}$ to $\mathbb{D}$ so is an automorphism. Clearly the two fundamental domains are normal in each other, and so the covering is a regular covering.

By following example 3.18 we can construct a general branched covering. Let $X$ be a Riemann surface and let $B$ be a discrete set in $X$. Let $X^* = X - B$ and suppose that $Y^*$ is the corresponding covering surface such that $f : Y^* \to X^*$ is a smooth covering of $X^*$. Then if $b_0 \in B$, let $\Delta \subset X$ be a parametric disc centered at $b_0$ so that $\Delta \bigcap B = \{b_0\}$. Then $f^{-1}(\Delta \backslash \{b_0\})$ is a collection of coverings of $\Delta \backslash \{b_0\}$.

For each component of $f^{-1}(\Delta \backslash \{b_0\})$ we add a point to $Y^*$ above $b_0$ and extend $f$ to map these points to $b_0$.

We do this for each point $b_i \in B$ and let $Y = Y^* \bigcup \{\text{Points added above } B\}$ and let $f^e : Y \to X$ be $f$ extended. Then $f^e$ is a holomorphic map branched only above $B$.

Similar with smooth coverings we can define the group of deck transformation for a branched covering $f : Y \to X$. As above we let $X$ be a compact Riemann surface, $B$ a finite set $\{b_1, \ldots, b_s\}$, $X^* = X \backslash B$. Let $D$ be a normal subgroup of $\pi_1(X^*, x_0)$, $x_0 \in X^*$, and let $Y^*$ be the corresponding regular cover of $X^*$ such that the covering $f : Y^* \to X^*$ is smooth. Then by extending $Y^*$ to $Y$, $f$ to $f^e$ and $G(Y^*/X^*)$ to $G(Y/X)$, we get that $G(Y/X)$ is isomorphic to $\pi_1(X^*, x_0)/D$.

Moreover, let $Y$ be a Riemann surface with a finite group of automorphisms $G$ and denote the set of orbits of $G$ by $Y/G$. Then if the only element of $G$ with a fixed point is the identity element, then $Y/G$ is made a Riemann surface $X$ by giving it the quotient topology and requiring that the natural map $f : Y \to Y/G = X$ is holomorphic. In fact, $f$ is a smooth covering.

For, suppose that $G$ admits non-identity elements with fixed points. Then for $y \in Y$ let $Stab(y) = \{g \in G : g(y) = y\}$, the stabilizer of $y$. For any $h \in G$ we have that the stabilizer of $h(y)$ is $Stab(h(y)) = h\ Stab(y)\ h^{-1}$.Hence the fixed points of non-identity elements of $G$ are orbits of $G$.

Let $Y^* = \{y : stab(y) = e\}$. Then $G \backslash \{e\}$ is fixed point free on $Y^*$. Let $X^* = Y^*/G$. By previous discussion we can extend the map $f : Y^* \to X^*$ to $f^e : Y \to X$, by adding to $X$ points corresponding to the orbits of points in $Y$ with stabilizers larger than just the identity element. $f^e$ is now holomorphic and branched precisely over these points (i.e. the points $X \backslash X^*$).

This shows the following.

**Theorem 3.20.** *If $G$ is a finite group of automorphisms on a Riemann surface $Y$, then the orbit space $Y/G$ is naturally a Riemann surface so that the projection $f : Y \to Y/G$ is a holomorphic map. The branching of $f$ occurs precisely at the fixed points of non-identity elements of $G$, and the number of sheets in the covering $Y \to Y/G$ is the order of $G$.*

Later in chapter 5, branched coverings will appear as projection maps given by quotients under the action of automorphism groups of the hyperbolic plane. These groups of automorphism are called *Fuchsian Groups* .

Now let $f : Y \to X$ be a normal covering of $n$ sheets, and suppose the ramification occurs above $B = \{b_1, \ldots, b_s\} \subset X$. Also let $\gamma_i$ be a path around $b_i$ so that

$$\pi_1(X \setminus B, x_0) = \{\gamma_1, \gamma_2, \ldots, \gamma_s : \gamma_1\gamma_2 \ldots \gamma_s = e\}.$$

Then, if $v_i$ is the ramification over a single point over $b_i$, for each path we have $\gamma_i^{v_i} = e$ and with

$$\varphi : \pi_1(X \setminus B, x_0) \to \Sigma_n$$

as the monodromy representation, the monodromy group, $M(Y/X)$, is generated by $\varphi(\gamma_1), \dots, \varphi(\gamma_s)$.

By denoting $\varphi(\gamma_i) = a_i \in \Sigma_n$, each $a_i$ satisfying $a_i^{v_i} = e$ and $a_1 a_2 \dots a_s = e$, we can now represent the monodromy group with the following definition.

**Definition 3.21.** A finite group generated by elements $a_1, \dots, a_s$ such that $a_1^{v_1} = \dots = a_s^{v_s} e$ and $a_1 a_2 \dots a_s = e$, will be denoted by $(v_1, \dots, v_s)$.

By the discussion before the definition the monodromy group $M(Y/X)$ is a $(v_1, \dots, v_s)$ group, and since $M(Y/X)$ is isomorphic to $G(Y/X)$ , the group of deck transformations, so is $G(Y/X)$. One example of how we can use this representation is when working with for example the triangle groups. However, before we can make use of this we need a little bit more theory.

**The Riemann-Hurwitz formula**

Let $f : Y \to X$ be an n-sheeted branched covering of $X$ with branch point set $B = \{b_1, \dots, b_s\}$, a finite subset of $X$. Then for each $b_i \in B$, let $\gamma_i$ be a path around $b_i$. From what we have seen previously, $\varphi(\gamma_j)$ is a permutation of length $m_i$.

The Riemann-Hurwitz formula (for branched coverings) is then

$$2g_Y - 2 = n(2g_X - 2) + \sum_{i=1}^{s} n(1 - \frac{1}{v_i}) = n(2g_x - 2 + s + \sum_{i=1}^{s} \frac{1}{v_k}) \tag{3.1}$$

This shows that the Euler characteristic is multiplicative by branched coverings.

**Example 3.22.** As a consequence of the theory, if $f : Y \to X$ is a branched covering of finite degree, where $X$ contains singularities and $Y$ may or may not contain singularities. Then by considering the universal covering of $X$ (which is also the universal covering of $Y$) we get the following picture.

Figure 9

Since $Y$ is an $n$-sheeted covering space of $X$ we have the following relation for the Euler characteristics of the two surface

$$\chi(Y) = n\chi(X).$$

Note that $\chi(X)$ will not in general be an integer, but is always fractional.

If we denote the order at each singular point by $m_i$ we then get the Riemann-Hurwitz formula

$$2-2g_Y = n(2-2g_X - \sum_{i=1}^{s}(1-\frac{1}{m_i}))$$

*Remark* 3.23. This construction was done by Riemann for meromorphic complex functions.

As a demonstration of the power of the Riemann-Hurwitz formula we can show the following two examples.

**Example 3.24.** We will show that a Riemann surface of genus 2 does not admit an automorphism of order 7. To see this let $X$ be a Riemann surface of genus 2 and let $f : X \to X$ be an automorphism of order 7. Then the group generated by $f$, $G =< f >$, together with the projection $\pi : X \to X/G$ gives us a 7-sheeted branched covering. The Riemann-Hurwitz formula 3.1 now gives us

$$2g_X - 2 = 7(2g_{X/G} - 2) + \sum_{i=1}^{s} 7(1-\frac{1}{m_i})$$

but since the automorphism is of order 7, every singular point must be of order 7 so this becomes

$$2g_X - 2 = 7(2g_{X/G} - 2) + \sum_{i=1}^{s} 7(1-\frac{1}{7})$$

now with $g_X = 2$ this becomes

$$2\cdot 2 - 2 = 7(2g_{X/G} - 2) + s7(1-\frac{1}{7}).$$

Hence we get the equation

$$14g_{X/G} + 6s = 16 \Longleftrightarrow 7g_{X/G} + 3s = 8$$

So if $g_{X/G} = 1$ we get that either $s = 0$ and

$$7 + 3\cdot 0 \neq 8$$

or $s = 1$ and

$$7 + 3 \neq 8.$$

So then if $g_{X/G} = 0$ we only get that

$$3s = 8$$

which is impossible, since $s \in \mathbb{Z}$.

*Remark* 3.25. As a consequence of the previous example, a surface of genus 2 cannot be a branched covering of the sphere with 3 singularities of orders 2,3 or 7. Hence the order of $Aut(X_{g_2}) < 84$.

**Example 3.26.** Similarly we can show that a Riemann surface of genus 3 does not admit an automorphism of order 5. With the same notation as above the Riemann-Hurwitz formula gives us

$$2g_X - 2 = 5(2g_{X/G} - 2) + \sum_{i=1}^{s} 5(1-\frac{1}{m_i})$$

and with $g_X = 2$ this becomes

$$2\cdot 3 - 2 = 5(2g_{X/G} - 2) + s5(1-\frac{1}{5}).$$

With some calculations we arrive at the equation

$$10g_{X/G} + 20s = 14 \iff 5g_{X/G} + 10s = 7$$

and clearly this equation doesn't have any solutions for $g_{X/G}, s \in \mathbb{Z}$.

**More on the monodromy representation**

By following the theory of monodromy representation, using the notation from definition 3.21, we can now investigate the types of groups that arises in some coverings.

Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a normal covering of $n$ sheets ($\mathbb{P}^1$ is the projective line but can also be viewed as the Riemann sphere). That is, the covering is an automorphism of a surface of genus zero. The Riemann-Hurwitz formula for the covering becomes

$$-2 = -2n + \sum_{i=1}^{s} n(1 - \frac{1}{v_i}) = n(-2 + s + \sum_{i=1}^{s} \frac{1}{v_k}) \tag{3.2}$$

and so $-2 + s + \sum_{i=1}^{s} \frac{1}{v_k}$ must be negative and equal to $-\frac{2}{n}$. The length of the permutations $v_i$ must be greater or equal to 2, hence the number of branching points, $s$, must be less than or equal to 3.

Now if there are only two branch points, $s = 2$, we have the case as mentioned earlier (before picture 2) that $G(\mathbb{P}^1/\mathbb{P}^1)$ is an $(n, n)$, a cyclic group of order $n$.

If $s = 3$, there are 4 possible cases, which all can be worked out by studying the only possible values for the $v_i$'s in formula 3.2.

(1) $G(\mathbb{P}^1/\mathbb{P}^1)$ is an $(2, 2, n)$, a dihedral group of order $2n$.
(2) $G(\mathbb{P}^1/\mathbb{P}^1)$ is an $(2, 3, 3)$, an alternating group of order 12.
(3) $G(\mathbb{P}^1/\mathbb{P}^1)$ is an $(2, 3, 4)$, a symmetric group of order 24.
(4) $G(\mathbb{P}^1/\mathbb{P}^1)$ is an $(2, 3, 5)$, an alternating group of order 60.

Since when $s = 3$ we have 3 branch points on the surface, these groups are sometimes denoted triangle groups. Their covering spaces are tessellations formed by the triangle with corners in the branch points.

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## 4. **Hyperbolic Geometry**

Since the Fuchsian groups considered in the next section, as mentioned earlier, are subgroups of the group of automorphisms of the hyperbolic plane there is a need to introduce some properties and definitions of hyperbolic geometry.

Unlike the Euclidian plane and the 2-sphere, the hyperbolic plane is hard to picture so instead we will use two different types of models of the hyperbolic plane. It is not difficult to see that the two models are equivalent to each other. One model is the upper half-plane defined by

$$\mathbb{H}^2 = \{x + iy : y > 0\}$$

which is supported by a metric derived from the differential $ds = \frac{|dz|}{Im[z]}$.

Another model is in terms of the unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

together with the metric derived from the differential $ds = \frac{2|dz|}{1-|z|^2}$.

The benefit of these two models is that *the circle of points at infinity* is easily defined. In the first case it is the real axis together with the points at infinity, where as in the second case it is simply the unit circle. Hence we can now establish the geodesics in both models.

In $\mathbb{H}^2$ there are two types of geodesics. We have all the straight lines parallel to the imaginary axis and we also have all half circles centered on the real axis.(For a complete development of this see Scott [13]).

In $\Delta$ the geodesics are given by circles intersecting the unit disc at straight angles. This includes the straight lines passing through the center of the unit circle (circles of infinite radius).

In order to describe the elements of Fuchsian groups the introduction of a special type of lines called the *pencils* is necessary. There are three different types of pencils in the hyperbolic plane. These are the *parabolic pencils*, the *elliptic pencils* and the *hyperbolic pencils*.

(1) Let $L$ and $L'$ be parallel geodesics with common endpoint $w$. The parabolic pencils $\mathcal{P}$ are defined to be the family of all geodesics with endpoint $w$. If we use the upper half plane, $H^2$ as a model with $w = \infty$, the geodesics in $\mathcal{P}$ are the lines $x = constant$.

FIGURE 10. Parabolic pencils

(2) Let $L$ and $L'$ be geodesics which intersect at the point $w$ in the hyperbolic plane. Then the elliptic pencils are defined to be the family of all geodesics through $w$.

FIGURE 11. Elliptic pencils

(3) Let $L$ and $L'$ be disjoint geodesics with $L_0$ as the common orthogonal geodesic. Then the hyperbolic pencils are defined to be the family of all geodesics which are orthogonal to $L_0$.

FIGURE 12. Hyperbolic pencils

Calculating the area of triangles and polygons, in the hyperbolic plane, will be of interest later in the thesis so we give two theorems for calculating the area without proofs. The proof for this is found in for example Beardons book [3].

**Theorem 4.1.** *(Gauss-Bonnet) For any triangle $T$ in the hyperbolic plane with interior angles $\alpha$, $\beta$ and $\gamma$ the hyperbolic area is given by*

$$area_h(T) = \pi - (\alpha + \beta + \gamma).$$

**Corollary 4.2.** *The angle sum of a hyperbolic triangle is less than $\pi$.*

**Theorem 4.3.** *If $P$ is any polygon in the hyperbolic plane with interior angles $\theta_1, \ldots, \theta_n$ then the hyperbolic area is given by*

$$area_h(P) = (n-2)\pi - (\theta_1 + \ldots + \theta_n).$$

**The Automorphisms of the hyperbolic plane**

The automorphisms of the hyperbolic plane are on the form

$$g(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. By viewing the elements in the group of such automorphisms, $Aut(\mathbb{H})$, as two by two matrixes, we see that this group is in fact $PSL(2, \mathbb{R})$, which acts as a group of isometries on the upper half-plane.

The conjugacy classes of $PSL(2, \mathbb{R})$ are given by the trace of the corresponding matrix, where an element $g \in PSL(2, \mathbb{R})$ is

(1) elliptic, if $trace(g) < 2$.
(2) parabolic, if $trace(g) = 2$.
(3) hyperbolic, if $trace(g) > 2$.

Another important result is that the hyperbolic metric (when using the model of the upper half-plane) is invariant under transformation of $PSL(2, \mathbb{R})$.

In other words, if $\gamma : [0, 1] \to \mathbb{H}^2$ is a piecewise differentiable path with $\gamma(t) = x(t)+iy(t) = z(t)$, its length given by

$$h(\gamma) = \int \frac{|dz|}{Im(z)} = \int_0^1 \frac{|\frac{dz}{dt}|}{y(t)} dt, \tag{4.1}$$

we can state the following theorem.

**Theorem 4.4.** *If $T \in PSL(2, \mathbb{R})$ then $h(T(\gamma)) = h(\gamma)$. That is, the hyperbolic length is invariant under transformations of elements in $PSL(2, \mathbb{R})$.*

*Proof.* Let

$$T(z) = \frac{az+b}{cz+d} \ (a, b, c, d \in \mathbb{R}, ad - bc = 1)$$

Then

$$\frac{dT}{dz} = \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

Also, if $z = x + iy$ and $T(z) = u + iv$, then

$$v = \frac{y}{|cz+d|^2},$$

and so

$$|\frac{dT}{dz}| = \frac{v}{y}.$$

Thus

$$h(T(\gamma)) = \int_0^1 \frac{|\frac{dT}{dt}|}{v} dt = \int_0^1 \frac{|\frac{dT}{dz}\frac{dz}{dt}|}{v} dt = \int_0^1 \frac{v|\frac{dz}{dt}|}{vy} dt = \int_0^1 \frac{|\frac{dz}{dt}|}{y} dt = h(y)$$

□

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## 5. **Fuchsian Groups and the Poincarè Theorem**

We will start by stating Poincarè's theorem in a somewhat informal structure in order to understand the reason of developing the theory.

Poincarè's theorem simply states that given a polygon $P$ and a collection of side pairing maps we can generate a group $G$ which is discrete and such that $P$ is a fundamental domain for $G$.

This result shows that $G$ is Fuchsian and we will see that the action of $G$ on the polygon $P$ tessellates the hyperbolic plane (i.e. the unit disc $\Delta$ or the upper half plane $H^2 = \{z \in \mathbb{C} : Im(z) > 0\}$). Hence we have found a developed (un-ramified) surface from a surface with ramified points.

Before we define Fuchsian groups we have to give a definition of discontinuous groups:

**Definition 5.1.** Let $X$ be any topological space and $G$ a group of homeomorphisms of $X$ onto itself. Then, $G$ acts *discontinuously on* $X$ if and only if for every compact subset $K$ of $X$,

$$g(K) \bigcap K = \varnothing,$$

except for a finite number of $g$ in $G$.

**Definition 5.2.** If a group $G$ acts discontinuously on a space $X$, then the stabilizer of any point of $X$ is finite. If all such stabilizers are trivial then we say that $G$ acts freely on $X$.

If $G$ acts freely and discontinuously on a surface $X$, then the natural map $X \to X/G$ is a covering map with covering group $G$. We are now ready to take on some examples.

**Example 5.3.** The map from the complex plane to the torus given by $\mathbb{C} \to \mathbb{C}/G$, where $G$ is the group generated by two parabolic elements $\tau_1$ and $\tau_2$. Then clearly $G$ acts discontinuously on $\mathbb{C}$ and also any stabilizer of any point is trivial. Hence we can say that $G$ acts freely on $\mathbb{C}$.

**Definition 5.4.** A group $G$ is a *Fuchsian group* if and only if there is some $G$-invariant disc in which $G$ acts discontinuously.

Fuchsian groups are discrete groups of hyperbolic isometries and their quotient spaces are Riemann surfaces. Fuchsian groups are similar to lattices, which are discrete groups of Euclidian isometries. The elliptic functions are invariant under lattices and in the case with Fuchsian groups we have a similar family of functions called the *automorphic functions*, which are invariant under Fuchsian groups.

The elements of a Fuchsian group can either be *parabolic*, *elliptic* or *hyperbolic*[5].

(1) A parabolic element, $g$, is of the form is of the form $g = \sigma_2\sigma_1$ where $\sigma_j$ is the reflection in the geodesic $L_j$ and where $L_1$ and $L_2$ determine a parabolic pencil.
(2) An elliptic element, $g$, is of the form $g = \sigma_2\sigma_1$ where $\sigma_j$ is the reflection in the geodesic $L_j$ and both $L_1$ and $L_2$ lie in an elliptic pencil.
(3) A hyperbolic element, $g$, is of the form $g = \sigma_2\sigma_1$ where $\sigma_j$ is the reflection in the geodesic $L_j$ and where $L_1$ and $L_2$ determine a hyperbolic pencil.

### **Fundamental Domains**

For a Fuchsian group $G$ acting on the hyperbolic plane $\Delta$ a *fundamental set* is a subset $F$ of $\Delta$ which contains only one element from each orbit in $\Delta$. Hence the action of $G$ on $F$ covers the hyperbolic plane i.e.

[5] see Beardon [3] chapter 7.

$$\bigcup_{f\in G} f(F) = \Delta$$

**Definition 5.5.** A subset $D$ of the hyperbolic plane is a *fundamental domain* for a Fuchsian group $G$ if and only if

(1) $D$ is a domain.
(2) there is some fundamental set $F$ such that $D \subset F \subset \overline{D}$.
(3) $area_h(\partial D) = 0$.

Clearly, if $D$ is a fundamental domain for a Fuchsian group $G$ then for all $g \in G$ except the identity we have that

$$g(D) \bigcap D = \varnothing$$

and also

$$\bigcup_{f\in G} f(\widetilde{D}) = \Delta$$

where $\widetilde{D}$ is the closure of $D$ with respect to the upper half-plane $\mathbb{H}^2$.

Now a natural question to ask is weather we can use the information from a group $G$ and it's fundamental domain to calculate the area of a fundamental domain corresponding to a subgroup of $G$. As it turns out, so is the case.

For simplicity, let $\mu$ denote the $area_h$. Then since $D \subset F \subset \widetilde{D}$ and $\mu(\partial D) = 0$ we must have that $\mu(D) = \mu(F)$, so we can either work with the fundamental domain or the fundamental set.

Next, suppose that $G$ is a Fuchsian group with fundamental set $F$, and that $F_1$ is another fundamental set for $G$. Then, since $\mu$ is invariant under each isometry we get that $F \subset \bigcup_{g\in G} g(F_1)$ and so

$$\begin{aligned}
\mu(F) &= \mu(F \bigcap [\bigcup_{g\in G} g(F_1)]) \\
&= \sum_{g\in G} \mu(F \bigcap g(F_1)) \\
&= \sum_{g\in G} \mu(g^{-1}(F) \bigcap F_1) \\
&= \sum_{g\in G} \mu(F_1 \bigcap g^{-1}(F)) \\
&= \mu(F_1 \bigcap [\bigcup_{g\in G} g^{-1}(F)]) \\
&= \mu(F_1)
\end{aligned}$$

Hence the area of a fundamental set (and also domain) is only dependent of the group which generates it. Now if $G_0$ is a subgroup of $G$ of index $k$ then we can write $G$ as a disjoint union of cosets $G = \bigcup_{n=1}^{k} G_0 g_n$, and we define a set (not necessarily a fundamental set) by

$$F^* = \bigcup_{n=1}^{k} g_n(F)$$

Now if $w \in \Delta$, then $g(w) \in F$ for some $g \in G$ and $g^{-1} = h^{-1}g_n$ for some $n$ and $h \in G_0$. Hence $h(w) \in g_n(F)$ which means that $F^*$ contains at least one point of each orbit of $G_0$ and so contains a fundamental set for $G_0$.

Now we need do see what happens with points in $F^*$ that lie in the same $G_0$-orbit, so suppose that $z$ and $f(z)$ are points in $F^*$, $f \in G_0$, and $z$ is not a fixed point of $G_0$. Then for some $m$ an $n$, the points $g_n^{-1}(z)$ and $g_m^{-1}(f(z))$ lie in $F$ and since $F$ doesn't contain two element of the same orbit we must have that $g_n g_m^{-1} f$ fixes $z$. Hence

$$g_m g_n^{-1} = f \in G_0$$

and so $G_0 g_m = G_0 g_n$ which implies that $n = m$. Thus $f$ fixes $z$ and so $f = I$ the identity map.

This shows that $F^*$ contains exactly one point from each orbit not containing any fixed points and at least one point from each obit of fixed points. By deleting the countable set of fixed points from $F^*$, the resulting set is a fundamental set for $G_0$ and since the areas of the fundamental sets of $G_0$ are equal we have that

$$\mu(F^*) = \mu(F_0)$$

Now, $F$ intersects an image of itself in at most a countable set of fixed points so

$$\begin{aligned}\mu(F^*) &= \sum_{n=1}^{n=k} \mu(g_n(F)) \\ &= k\mu(F).\end{aligned}$$

Notice that the area for a fundamental domain of a subgroup of index $k$ is $k$ times larger than the fundamental domain of the group.These results can be gathered in the following theorem

**Theorem 5.6.** *If $G$ is a Fuchsian group with a fundamental domain $F$ and $G_0$ is a subgroup of index $k$ of $G$ with corresponding fundamental domain $F_0$ then*

$$area_h(F_0) = k \cdot area_h(F)$$

This is in fact just the Riemann-Hurwits formula, To using the covering map $\mathbb{H}/G_0 \to \mathbb{H}/G$.

When working with fundamental domains, there is a condition we must have in order to avoid that the quotient of the fundamental domain and the group is not compact. If the quotient is not compact then it will not be homeomorphic to the quotient $\Delta/G$. (For a nice example of this see Beardon [3] pages 206-207).

Before we can state the condition we will start by looking at the relationship between the two quotients $\widetilde{D}/G$ and $\Delta/G$.

Let $G$ be a Fuchsian group acting on $\Delta$ and let $D$ be a fundamental domain for $G$ in $\Delta$. On one hand we have the natural continuous and open projection

$$\pi : \Delta \to \Delta/G.$$

On the other hand, there is a continuous projection

$$\widetilde{\pi} : \widetilde{D} \to \widetilde{D}/G,$$

where $\widetilde{D}/G$ inherits the quotient topology by identifying equivalent points on the boundary of $D$.

Now our aim is to give the conditions for when $\Delta/G$ and $\widetilde{D}/G$ are topologically equivalent, that is when there is a homeomorphism between them.

The elements of $\Delta/G$ are the orbits $G(z)$ whilst the elements of $\widetilde{D}/G$ are the sets $\widetilde{D} \bigcap G(z)$. Also

$$\pi(z) = G(z) \text{ and } \widetilde{\pi}(z) = \widetilde{D} \bigcap G(z).$$

Next, let $\tau : \widetilde{D} \to \Delta$ denote the inclusion map. We now construct a map $\theta$ between the two quotient groups by

$$\theta : \widetilde{D} \bigcap G(z) \to G(z).$$

$\theta$ is well defined since for each $z$, $\widetilde{D} \bigcap G(z) \neq \varnothing$ and $\theta\widetilde{\pi} = \pi\tau$.

The last fact means that the following diagram commutes

$$\begin{array}{ccc} \widetilde{D} & \xrightarrow{\tau} & \Delta \\ \widetilde{\pi} \downarrow & & \downarrow \pi \\ \widetilde{D}/G & \overrightarrow{\sigma} & \Delta/G \end{array}$$

To gather up some information we give the following properties of the above maps

(1) $\theta$ and $\tau$ are injective.
(2) $\pi$, $\widetilde{\pi}$ and $\theta$ are surjective.
(3) $\pi$, $\widetilde{\pi}$ and $\tau$ are continuous.
(4) $\pi$ is an open map.

We would like to have that $\theta$ is also a continuous map, and as it turns out, so is the case. In order to see this, let $U$ be an open subset of $\Delta/G$, then by the commutativity of the above diagram we get that

$$\widetilde{\pi}^{-1}(\theta^{-1}(U)) = \widetilde{D} \bigcap \pi^{-1}(U)$$

and the right part is open in $\widetilde{D}$ as $\pi$ is continuous. Now for any subset $V$, $\widetilde{\pi}^{-1}(V)$ is open in $\widetilde{D}$ if and only if $V$ is open in $\widetilde{D}/G$. Thus $\theta^{-1}$ is open in $\widetilde{D}/G$ and hence $\theta$ is continuous.

We can now state a the condition we need in order to assure that the fundamental domain behaves nicely.

**Definition 5.7.** A fundamental domain $D$ for $G$ is said to be *locally finite* if and only if each compact subset of $\Delta$ meets only finitely many $G$-images of $\widetilde{D}$.

Stated in another way, this result becomes: If $D$ is locally finite, each $z$ has a compact neighborhood $N$ and an associated finite subset $g_1, ...g_n$ of $G$ with

(1) $z \in g_1(\widetilde{D}) \bigcap ... \bigcap g_n(\widetilde{D})$
(2) $N \subset g_1(\widetilde{D}) \bigcup ... \bigcup g_n(\widetilde{D})$
(3) $h(D) \bigcap N = \varnothing$ unless $h$ is some $g_j$.

When working with fundamental domains there are some special domains that are natural to examine closely. Those are the domains which are polygons. These polygons are hyperbolic polygons so they may have sides and vertices on the line at infinity.

**Definition 5.8.** Let $G$ be a Fuchsian group. Then $P$ is a *convex fundamental polygon* if and only if $P$ is a convex, locally finite fundamental domain for $G$.

If we consider any $g \in G$, not the identity, then clearly $\widetilde{p} \bigcap g(\widetilde{P})$ is convex. Also $\widetilde{p} \bigcap g(\widetilde{P})$ cannot contain three non-collinear points, since it then would mean that $P \bigcap g(P) \neq \varnothing$. Hence $\widetilde{p} \bigcap g(\widetilde{P})$ must be a geodesic segment (possibly empty). So we can give the definition of the sides and vertices of $P$ by the following:

**Definition 5.9.** A side of $P$ is a geodesic segment of the form $\widetilde{p} \bigcap g(\widetilde{P})$ of positive length. A vertex of $P$ is a single point of the form $\widetilde{p} \bigcap g(\widetilde{P}) \bigcap h(\widetilde{P})$ for distinct $I$, $g$ and $h$.

Since $G$ is countable and only finitely many images of $\widetilde{P}$ can meet any compact subset of $\Delta$, $P$ has only countably many sides and only finitely many vertices and sides can meet any given compact subset of $\Delta$. Also, since the sides and vertices lie inside $\partial P$, $\partial P$ is the union of the sides of $P$.

We see that in fact all the properties expected from a Euclidian convex fundamental polygon holds with this definition.

Now for the side pairing elements of $G$, we let $G^*$ be the elements of $G$ such that $\widetilde{p} \bigcup g(\widetilde{P})$ is a side of $P$ and denote the set of sides by $S$. So each $g \in G$ determines a unique side $s \in S$. Hence there is a surjective map

$$\Phi : G^* \to S$$

given by

$$\Phi(g) = \widetilde{p} \bigcup g(\widetilde{P}).$$

In fact, $\Phi$ is a bijection for if $\Phi(g) = \Phi(h)$ then we must have that $g(\widetilde{P}) = h(\widetilde{P})$ and this can only occur for sides if $g = h$.

The existence of the inverse $\Phi^{-1}$ shows that to each side $s$ there is a unique $g_s$ in $G^*$ such that

$$s = \widetilde{p} \bigcup g_s(\widetilde{P}).$$

So the inverse gives that

$$g_s^{-1}(s) = \widetilde{p} \bigcup g_s^{-1}(\widetilde{P}) = s'$$

which is also a side. So we can get other sides by using the rule $g_s^{-1}(s) = s'$ and so we have constructed a map $s \to s'$ of $S$ onto itself, a *side pairing map* of $P$.

The next theorem is useful for calculating the Fuchsian group when given a fundamental polygon. For the proof the reader is referred to Beardon [3].

**Theorem 5.10.** *The side paring elements $G^*$ of $P$ generate $G$.*

Now we are ready to state the main theorem of this chapter, the Poincaré Theorem.

**Theorem 5.11.** *For a polygon $P$ with a side pairing $\Phi$ that satisfies*

(1) *For every vertex $x$ of $P$ there are vertices $x_0, x_1, \dots, x_n$ of $P$ and elements $f_0, f_1, \dots, f_n$ of $G$ such that for the sets $N_j = \{y \in \widetilde{P} : d(y, x_j) < \epsilon\}$ the sets $f_j(N_j)$ are non-overlapping sets whose union is $B(x, \epsilon) = \{y \in \Delta : d(x, y) < \epsilon\}$ and such that each $f_{j+1}$ is of the form $f_j g_s$ for some $s$ ($j = 1, \dots, n; f_{n+1} = I$).*

(2) *The $\epsilon$ can be chosen independently of $x$ in $\widetilde{P}$.*

*$G$ is discrete and $P$ is a fundamental polygon for $G$.*

*Remark* 5.12. The first property in the theorem simply tells us that when identifying the sides of the polygon by the side pairing we expect to obtain a smooth surface.

*Proof.* We will start by constructing a space $X^*$ which is tesselated by the group action. Then we will identify this tessellation with the $G$-images of the polygon $P$ in the original space.

For the first part, let $X$ be a non-empty set and suppose that $P$ is a polygon in $X$. Note that since we have a side pairing $\Phi$ of $P$, this means that there is a self-inverse map $s \mapsto s'$ of the sides of $P$ onto itself and associated with each pair $(s, s')$ there is a bijection $g_s$ of $X$ onto itself with

$$g_s(s) = s'$$

and

$$g_{s'} = (g_s)^{-1}.$$

Now let $G$ be the group generated by these side pairing elements, and form the Cartesian product $G \times \widetilde{P}$. In a way, $G \times \widetilde{P}$ can be thought of as a collection of copies $(g, \widetilde{P})$ of $\widetilde{P}$ which are just indexed by $G$. It can be shown[6] that one can define an equivalence relation on $G \times \widetilde{P}$, where the equivalence class containing $(g, x)$ is denoted $< g, x >$, such that if

$$< g, x >=< h, y >$$

then

$$g(x) = h(y)$$

and

$$< fg, x >=< fh, y > \text{ for } f \in G$$

Now each $f \in G$ induces a map $f^* : X^* \to X^*$ given by

$$< g, x >\mapsto< fg, x >.$$

Note that we have the two relations

$$(f^{-1})^* = (f^*)^{-1}$$

and

$$(fh)^* = f^*h^*.$$

So the set of all such $f^*$ form a group $G^*$ of bijections of $X^*$ onto itself and $f \mapsto f^*$ is a homomorphism of $G$ onto $G^*$.

If we now define

$$< P >= \{< I, x >: x \in P\}$$

and similar for $\widetilde{P}$ we get that the action of $G^*$ on $< P >$ tessellates $X^*$ in the sense that

$$\bigcup_{g^*} g^* < \widetilde{P} >= X^* \tag{5.1}$$

and whenever $g^* \neq h^*$ we have

$$g^* < P > \bigcap h^* < P >= \varnothing. \tag{5.2}$$

With this tessellation we can now easily construct a natural map $\alpha : X^* \to X$ by the relation

$$\alpha < g, x >= g(x).$$

**Lemma 5.13.** (1) *If $\alpha$ is surjective, then*

$$\bigcup_{g \in G} g(\widetilde{P}) = X.$$

(2) *If $\alpha$ is surjective, then for distinct $g$ and $h$ in $G$,*

$$g(P) \bigcap h(p) = \varnothing$$

*Proof.* First note that the composition $\alpha g^* = g$ is true from the definition of $\alpha$.

[6] See Beardon [3] page 242-

(1) If $\alpha$ is surjective then clearly by the definition of $\alpha$, $\alpha(X^*) = X$, so by applying $\alpha$ to equation 5.1 we obtain

$$\alpha(\bigcup_{g^*} g^* < \widetilde{P} >) = \bigcup_{g \in G} g(\widetilde{P}) = X.$$

(2) By the injectivity of $\alpha$ it is easy to see that when applied to 5.2 we obtain the desired result:

$$\alpha(g^* < P >) \bigcap \alpha(h^* < P >) = g(P) \bigcap h(p) = \varnothing$$

□

One can show that $X^*$ is Hausdorff, connected and also that, for every $x^*$ in $X^*$ there is an open neighborhood $N^*$ such that the restriction of $\alpha$ to $N^*$ is a homeomorphism of $N^*$ onto an open subset of $X$. But for the proof of this we need to go through several steps using topology and this part of the proof is omitted here[7].

Now we restrict ourself to supposing that $(X, d)$ is the hyperbolic plane with the hyperbolic metric, that $P$ is a hyperbolic polygon and that $\Phi$ is some given set of side pairing isometries. We need now to show that $G$ is discrete and that $P$ is a fundamental polygon for $G$.

By the (2) property of $\Phi$ from the theorem we are ensured that each curve in $X$ can be lifted into a curve in $X^*$ and so $(X^*, \alpha)$ is a smooth unlimited covering surface of $X$ and $\alpha$ maps $X^*$ onto $X$. As $X$ is simply connected, the Monodromy Theorem tells us that $\alpha$ is a homeomorphism and from lemma 5.13 we get that $G$ is discrete by the definition. □

[7] See Beardon [3] for full details on this

Now to some results of Fuchsian groups. If $f : X \to Y$ is a branched covering surfaces, where $X = \mathbb{H}/\Gamma_1$ and $Y = \mathbb{H}/\Gamma$, then $\Gamma_1$ is a subgroup of $\Gamma$. Also, if $|\Gamma : \Gamma_1| < \infty$ then the epimorphism $\varphi : \Gamma \to \Sigma_{|\Gamma:\Gamma_1|}$ determines the covering $f : X \to Y$.

$\Gamma_1$ is a Fuchsian group if and only if $X$ is a Riemann surface.

The covering $f : X \to Y$ is regular if and only if $\Gamma_1 \trianglelefteq \Gamma$. The factor group $G = \Gamma/\Gamma_1$ determines the covering.

$$\theta : \Gamma \stackrel{epim.}{\to} G$$
$$\Gamma = ker\ \theta.$$

Let $X$ be a Riemann surface uniformized by $\Gamma$ without elliptic elements, so that $\Gamma$ acts freely on $\mathbb{H}$. Then the covering

$$\pi : \mathbb{H} \to X$$

is unbranched.

Let $G$ be a group of automorphisms of $X$ and $X/G$ is a Riemann surface then we have

Figure 13. Picture of coverings

So $X/G$ is uniformized by a Fuchsian group $\Lambda$.

$$\Gamma \trianglelefteq \Lambda \text{ and } \Lambda/\Gamma \simeq G.$$
$$(\theta : \Lambda \to G \text{ and } ker\ \theta = \Gamma)$$

Given a group of automorphisms, $\Lambda$, of $X$, the biggest group where $\Gamma$ is normal is the normalizer $N(\Gamma) \leqslant \Lambda$, so $N(\Gamma)$ is a Fuchsian group.

From theorem 5.6 we know that

$$\mu(\mathbb{H}/\Gamma) = |G|\mu(\mathbb{H}/\Lambda) \tag{5.3}$$

and the area of the fundamental domain in $\mathbb{H}$ generated by $\Lambda$ is given by the area formulas from chapter 4

$$\mu(\mathbb{H}/\Lambda) = (n-2)\pi - (\theta_1 + \ldots + \theta_n).$$

where $\theta_i$ are the angles at the vertices.

It is not difficult to see that the smallest area is when the fundamental domain is a triangle so the area then becomes

$$\mu(\mathbb{H}/\Lambda) = \pi - (\theta_1 + \theta_2 + \theta_3) = \pi(1 - \tfrac{1}{m_1} - \tfrac{1}{m_2} - \tfrac{1}{m_3}),$$

and $\mu(\mathbb{H}/\Lambda)$ obtains its smallest value for the angles $(m_1, m_2, m_3) = (2, 3, 7)$. Thus

$$\mu(\mathbb{H}/\Lambda) \leq \pi(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{7})$$

Because the covering $\pi : \mathbb{H} \to X = \mathbb{H}/\Gamma$ is unbranched by the Gauss-Bonnet formula we get that

$$\mu(\mathbb{H}/\Gamma) = \pi(2g_{\mathbb{H}/\Gamma} - 2 + \sum(1 - \frac{1}{m_i})) = \pi(2g_{\mathbb{H}/\Gamma} - 2).$$

Hence, by equation 5.3 we get that

$$\pi(2g_{\mathbb{H}/\Gamma} - 2) = |G|\pi(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{7})$$

and so

$$|G| \leq 84(g_{\mathbb{H}/\Gamma} - 1)$$

Groups $G$ with order exactly $84(g_{\mathbb{H}/\Gamma} - 1)$ are called Hurwitz groups. They are the largest automorphism groups of surfaces of genus $g$.

**Example 5.14.** A beautiful example of a Hurwitz group is $G = PSL(2,7)$ with order 168. $PSL(2,7)$ is the automorphism group of the Klein quartic, the surface of genus 3 ($168 = 84(3-1)$) with equation in homogenous coordinates

$$x^3y + y^3z + z^3w + w^3x = 0.$$

For now, note that the surface is in $\mathbb{P}^2\mathbb{C}$, but this will be more explained in the following chapter.

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## 6. **Riemann Surfaces and Algebraic Curves**

If we let $f(x, y)$ be a polynomial with real coefficients in two variables, then the graph in $\mathbb{R}^2$ defined by the equation $f(x, y) = 0$ is called a real algebraic curve and the degree of the curve is the degree of $f(x, y)$. However since the real number field is not algebraically closed we wish to generalize the notion of an algebraic curve over a field where we can obtain complete results.

Therefore we consider $f(x, y)$ as a polynomial with complex coefficients in two variables and consider the equation

$$f(x, y) = 0 \tag{6.1}$$

as an algebraic curve in $\mathbb{C}^2$.

The reason for this is seen when considering the equation solving the number of intersection points of a curve with a straight line. It is well known from real analysis that such equation doesn't have all solutions in the real field and so working in the complex field yields the desired result.

In fact, writing the line $L$ as a line passing through the origin with parametric equations

$$\begin{cases} x = \alpha t \\ y = \beta t \end{cases} \tag{6.2}$$

and writing $f(x, y)$ as

$$f(x, y) = f_n(x, y) + f_{n-1}(x, y) + \ldots + f_0$$

where each $f_i(x, y)$ is a homogeneous polynomial of degree $k$. By substituting 6.2 into 6.1 we get

$$f_n(\alpha, \beta)t^n + f_{n-1}(\alpha, \beta)t^{n-1} + \ldots + f_0 = 0 \tag{6.3}$$

which is the equation determining the intersection points of the complex straight line $L$ (6.2) with the complex algebraic curve $C$ (6.1). Now from the fundamental theorem of algebra, as long as $f_n(\alpha, \beta) \neq 0$, the equation (6.3) will have exactly $n$ roots (multiple roots counted with multiplicity). Hence the complex algebraic curve $C$ of degree $n$ intersects the complex line $L$ at $n$ points where the intersection points corresponding to multiple roots are thought of as multiple points of intersection.

However, if we would have that

$$f_n(\alpha, \beta) = f_{n-1}(\alpha, \beta) = \ldots = f_{m+1}(\alpha, \beta) = 0,$$
$$f_m(\alpha, \beta) \neq 0,$$

then $L$ and $C$ intersect in only $m$ points in $\mathbb{C}^2$. In this case the remaining $(n - m)$ points are regarded to be intersections at infinity. So we say that $L$ and $C$ have a point of intersection of multiplicity $(n - m)$ at infinity. Hence to be able to describe the complete algebraic curve we add the line at infinity to $\mathbb{C}^2$ and thus we have obtained the complex projective plane $P^2\mathbb{C}$.

One question remains to answer, and that is how we add the line at infinity to $\mathbb{C}^2$. It turns out that the most convenient way of doing this is by using homogeneous coordinates.

**Definition 6.1.** For a point $(x, y) \in \mathbb{C}^2$, its homogeneous coordinates are given by any set of complex numbers $(\zeta, \xi, \eta)$ satisfying

$$x = \xi/\zeta, y = \eta/\zeta. \tag{6.4}$$

Clearly, if $(\zeta, \xi, \eta)$ are homogeneous coordinates for $(x, y)$ then so are $(\lambda\zeta, \lambda\xi, \lambda\eta)$ for $\lambda \in \mathbb{C}\backslash\{0\}$. In order for (6.4) to be well defined it is necessary that $\zeta \neq 0$. However, if $\xi$ and $\eta$ are not both 0, then as $\zeta \to 0$ the points $x = \xi/\zeta$ and $y = \eta/\zeta$ approach infinity in the direction $\xi : \eta$. So by using this in homogeneous coordinates we can represent the line at infinity as the set of points $(0, \xi, \eta)$ in $\mathbb{C}^3$.

*Remark* 6.2. There is a rigorous way of defining the complex projective plane by introducing an equivalence relation $\sim$ on the set $\mathbb{C}^3\backslash\{0,0,0\}$ and then letting the complex projective plane be defined as the quotient space induced by the equivalence relation[8].

Now back to the algebraic curve, its representation in homogeneous coordinates given by (6.1) is then

$$F(\zeta,\xi,\eta) = f_n(\xi,\eta) + f_{n-1}(\xi,\eta)\zeta + \ldots + f_0\zeta^n = 0.$$

Here we have multiplied both sides by $\zeta^n$. Note that $F(\zeta,\xi,\eta)$ is a homogeneous polynomial in $(\zeta,\xi,\eta)$ and in general the equation

$$F(\zeta,\xi,\eta) = 0 \tag{6.5}$$

represents an algebraic curve in the complex projective plane, and the degree of $F(\zeta,\xi,\eta)$ is called the degree of this curve. We call the equation (6.5) the *homogeneous equation* of the curve. If we restrict ourselves to $\mathbb{C}^2 = P^2\mathbb{C}\backslash L_\infty$, then the curve satisfies the *affine equation*

$$f(x,y) = 0 \tag{6.6}$$

where

$$f(x,y) = F(1,x,y).$$

Hence the homogeneous equation of a curve determines the affine equation of the curve on $\mathbb{C}^2 = P^2\mathbb{C}\backslash L_\infty$ while on the other hand the degree of the curve $(n)$ and its affine equation $f(x,y) = 0$ determines its homogeneous equation

$$F(\zeta,\xi,\eta) = 0$$

where

$$F(\zeta,\xi,\eta) = \zeta^n f(\xi,\eta).$$

**Example 6.3.** Take the curve given by

$$x^3 - x^2 + y^2.$$

By changing into homogeneous coordinates we get

$$\frac{\xi^3}{\zeta^3} - \frac{\xi^2}{\zeta^2} + frac\eta\zeta^2 = 0.$$

So the homogeneous coordinate representation becomes

$$F(\zeta,\xi,\eta) = \xi^3 + (\eta^2 - \xi^2)\zeta = 0.$$

[8]See Griffith [8]

The real graph of this curve is:

FIGURE 14. The real graph of $x^3 - x^2 + y^2$.

**Definition 6.4.** If an algebraic curve $C$ is given by

$$F(\zeta, \xi, \eta) = 0$$

and $F$ can be written as a product of irreducible homogeneous polynomials

$$F = F_1^{m_1} \cdot F_2^{m_2} \cdot \ldots \cdot F_k^{m_k}$$

then we can write the curve as

$$C = m_1 C_1 + m_2 C_2 + \ldots + m_k C_k$$

where

$$C_j = \{(\zeta, \xi, \eta) \in P^2\mathbb{C} : F_j(\zeta, \xi, \eta) = 0\} \text{ for all } j = 1, 2, \ldots, k.$$

Each $C_j$ is called an *irreducible component* of $C$. In the case where $F$ is irreducible we call $C$ an *irreducible* curve.

**The Normalization Theorem**

The relation between the study of algebraic curves and compact Riemann surfaces is given by the following theorem called the normalization theorem.

**Theorem 6.5.** *For any irreducible algebraic curve $C \subset P^2\mathbb{C}$, there exists a compact Riemann surface $\widetilde{C}$ and a holomorphic mapping*

$$\sigma : \widetilde{C} \to P^2\mathbb{C}$$

*such that $\sigma(\widetilde{C}) = C$, and $\sigma$ is injective on the inverse image of the set of smooth points of $C$.*

The Riemann surface $\widetilde{C}$ together with the holomorphic map $\sigma$ is called the *normalization* of the curve $C$.

The proof of the theorem will be given later. We will first have to look at some of the properties needed for explaining the theorem.

We start by considering the plane algebraic curve given by

$$C = \{(z, x, y) \in P^2\mathbb{C} : F(z, x, y) = 0\}.$$

From the implicit function theorem we get that in the neighborhood of a smooth point, $C$ corresponds biholomorphically to an open set in $\mathbb{C}$. The points that will cause trouble are the singular points $p$ where we have that

$$\frac{\partial F}{\partial z}(p) = \frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = 0.$$

So we need to investigate what happens at these singularities.

Let $p$ be any point on the curve $C$ and choose coordinates of $P^2\mathbb{C}$ such that the Kummer coordinates of $p$ becomes $p = [1, 0, 0]$. Since the curve satisfies the homogeneous equation $F(z, x, y) = 0$ we know there is an affine equation such that $f(x, y) = 0$ where

$$f(x, y) = F(1, x, y).$$

So the curve corresponding to the affine equation is $C \bigcap \mathbb{C}^2$, where $\mathbb{C}^2$ is canonically embedded in $P^2\mathbb{C}$ as usual

$$\mathbb{C}^2 \to P^2\mathbb{C},$$
$$(x, y) \mapsto [1, x, y].$$

Writing $f(x, y)$ as a sum of homogeneous polynomials

$$f(x, y) = f_k(x, y) + f_{k-1}(x, y) + \ldots + f_d(x, y)$$

where $f_j(x, y)$, $(j = k, \ldots, d)$, is a homogeneous polynomial of degree $j$ and $f_k \not\equiv 0$. Since $f(0, 0) = 0$ we must have that $k \geq 1$.

If $k = 1$, then $f_1(x, y) = ax + by \not\equiv 0$, hence

$$\frac{\partial f(0,0)}{\partial x} = a \neq 0 \text{ or } \frac{\partial f(0,0)}{\partial y} = b \neq 0.$$

So $p = (0, 0)$ is a smooth point of $C$, and $C$ has a tangent line at $p$. We therefore call $p$ a *simple point* of $C$.

From this we see that for $p$ to be a singularity of $C$ we must have that $k \geq 2$. For $k = 2$ we get that

$$f_0 \equiv f_1 \equiv 0$$

and

$$f_2(x, y) = ax^2 + 2bxy + cy^2 \not\equiv 0.$$

In this case $C$ has two tangent lines given by the equation

$$f_2(x, y) = ax^2 + 2bxy + cy^2 = 0,$$

and we call $p$ a *double point* of $C$.

In general, if

$$f_0 \equiv f_1 \ldots \equiv f_{k-1} \equiv 0, \; f_k \not\equiv 0$$

then $C$ has $k$ tangent lines at $p$ (possibly multiple tangent lines) which are all given by the equation

$$f_k(x, y) = 0$$

and we call $p$ a *k-tuple point* of $C$.

**Definition 6.6.** $p$ is called an ordinary k-tuple point of $C$ if $p$ is a k-tuple point of $C$ and the k tangent lines at this point are distinct.

**Example 6.7.** By studying the curve $x^3 - x^2 + y^2 = 0$, we see that

$$f_0 \equiv f_1 \equiv 0.$$

But

$$f_2 = y^2 - x^2 = (y - x)(y + x)$$

so the curve has two tangent lines at the origin, namely

$$y = x \text{ and } y = -x.$$

**Example 6.8.** Another example is the curve given by $(x^2 + y^2)^2 + 3x^2y - y^3$. Clearly

$$f_0 \equiv f_1 \equiv f_2 \equiv 0.$$

And

$$f_3 = 3x^2y - y^3 = y(3x^2 - y^2) = y(\sqrt{3}x - y)(\sqrt{3}x + y).$$

In this case we see that the origin is an ordinary 3-tuple point with tangent lines given by

$$y = 0,\ y = \sqrt{3}x,\ y = -\sqrt{3}x$$

FIGURE 15. The real graph of $(x^2 + y^2)^2 + 3x^2y - y^3$.

Since we are interested in the singular points of $C$, denote the set of all singular points of $C$ by $S$. The following theorem shows that this set is finite.

**Theorem 6.9.** *An irreducible plane algebraic curve $C$ has at most finitely many singular points.*

*Proof.* (Outline of proof[9]).

The idea is to show that $S \bigcap \mathbb{C}^2$ is a finite set. Having done this and using the fact that the irreducible algebraic curve $C$ and the line at infinity $L_\infty$ can intersect in at most a finite number of points, hence $S \bigcap L_\infty$ contains at most a finite number of points, the result follows.

To show that $S \bigcap \mathbb{C}^2$ is a finite set, we need to state (without proofs) some algebraic theory about the eliminant and discriminant of polynomials.

**Lemma 6.10.** *Suppose $D$ is a unique factorization domain, and that*

$$f(x) = a_0x^m + \ldots + a_m \ (a_0 \neq 0,$$

$$g(x) = b_0x^n + \ldots + b_n \ (b_0 \neq 0,$$

*are polynomials over $D$. Then a necessary and sufficient condition for $f$ and $g$ to possess a nontrivial common factor is that it exist two polynomials $F, G \in D[x]$, not both equal to 0, which satisfy*

$$deg\ F < m,\ deg\ G < n,\ fG = gF.$$

**Theorem 6.11.** *Suppose $D$ is a U.F.D., and*

$$f(x) = a_0x^m + \ldots + a_m \ (a_0 \neq 0,$$

$$g(x) = b_0x^n + \ldots + b_n \ (b_0 \neq 0,$$

*are two polynomials in $D$. Then for $f$ and $g$ to have a nontrivial common factor, a necessary and sufficient condition is that the determinant $\mathcal{R}(f, g)$ of the below matrix is equal to 0.*

[9] For complete proof see [8]

$$\mathcal{R}(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & \cdots & \cdots & a_m & & & \\ & a_0 & a_1 & \cdots & \cdots & \cdots & a_m & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & a_0 & a_1 & \cdots & \cdots & \cdots & a_m \\ b_0 & b_1 & \cdots & \cdots & b_n & & & & \\ & b_0 & b_1 & \cdots & \cdots & b_n & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & b_0 & b_1 & \cdots & \cdots & b_n \end{vmatrix}$$

**Definition 6.12.** The determinant, $\mathcal{R}(f,g)$, is called the eliminant of $f$ and $g$.

**Corollary 6.13.** *Under the hypothesis in theorem 6.11 there exist polynomials $\alpha, \beta \in D[x]$, with deg $\alpha < n$, deg $\beta < m$ such that*

$$\alpha(x)f(x) + \beta(x)g(x) = \mathcal{R}(f,g).$$

**Definition 6.14.** Suppose $D$ is a *U.F.D.*. Then the eliminant of $f \in D[x]$ and its derivative polynomial $f' \in D[x]$, denoted by

$$\mathcal{D}(f) \equiv \mathcal{R}(f,f'),$$

is called the discriminant of $f$.

**Corollary 6.15.** *Suppose $D$ is a U.F.D.. Then a necessary and sufficient condition for $f \in D[x]$ to have multiple factors is that its discriminant be equal to 0.*

$$\mathcal{D}(f) = \mathcal{R}(f,f') = 0.$$

**Lemma 6.16.** *Suppose $C$ is a plane algebraic curve of degree $n$. Then it is possible to choose a coordinate system in $P^2\mathbb{C}$ such that $C$ possesses an affine equation of the following form:*

$$f(x,y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x) = 0,$$

*where $a_j(x) \in \mathbb{C}[x]$ with deg $a_j(x) \le j$, or $a_j(x) \equiv 0$.*

Now to finish the proof of Theorem 6.9 we know that we can choose a coordinate system such that the affine equation $f(x,y)$ of $C$ satisfies the conditions of Lemma 6.16. So consider $f$ as an element of $\mathbb{C}[x][y]$, and look at its discriminant

$$\mathcal{D}(f) = \mathcal{R}(f, \tfrac{\partial f}{\partial y}).$$

Since $S$ is the set of singular points of $C$ we must have that

$$S \bigcap \mathbb{C}^2 \subset \{(x,y) \in \mathbb{C}^2 : f(x,y) = \tfrac{\partial f}{\partial y}(x,y) = 0\},$$

and by Theorem 6.11, the projection of this set into the $x$-axis is

$$D_x = \{x \in \mathbb{C} : \mathcal{D}(f)(x) = 0\}.$$

This is the null set of a non-zero polynomial and therefore consists of only a finite number of points, and for each such $x_0 \in \mathbb{C}$ satisfying $\mathcal{D}(f)(x_0) = 0$, there can be only a finite number of $y$ such that

$$f(x_0, y) = 0.$$

Hence $S \bigcap \mathbb{C}^2$ is a finite set. □

The next thing we need to do in our investigation of the singularities $S$ of the plane algebraic irreducible curve $C$ is to check whether the sets $C^* = C \backslash S$ and $C$ are connected. If so is the case we can deduce from the implicit function theorem that $C^*$ is a one-dimensional complex manifold, i.e., a Riemann surface. In general, unless $C$ itself is smooth, $C^*$ is a non-compact Riemann surface.

To prove that $C^*$ and $C$ are connected we will use analytical continuation. An *analytic function element* is a pair, $(\Delta, f)$, which consists of an open disc $\Delta \in \mathbb{C}$ and an analytic function $f$ defined on this disc. Moreover, two analytic function elements $(\Delta_1, f_1)$ and $(\Delta_2, f_2)$ are said to be *direct analytic continuations* of each other if

$$\Delta_1 \bigcap \Delta_2 \neq \varnothing$$

and on $\Delta_1 \bigcap \Delta_2$ we have

$$f_1 \equiv f_2.$$

An *analytical chain* is a collection of analytic function elements $(\Delta_1, f_1), (\Delta_2, f_2), \ldots, (\Delta_N, f_N)$ in which any pair of successive analytic function element are direct analytic continuations of each other.

Now if $\gamma$ is a path in $\mathbb{C}$ starting at $a$ and ending at $b$ and suppose that $(\Delta_0, f_0)$ is an analytic function element satisfying $a \in \Delta_0$. We then say that $(\Delta_0, f_0)$ can be *analytically continued along the path* $\gamma$, if there exists a partition of $\gamma$

$$\gamma = \bigcup_{j=0}^{N} \gamma_j,$$

$$a = x_0 < x_1 < \ldots < x_{N+1} = b,$$

where $\gamma_j$ is the restriction of $\gamma$ to $[x_j, x_{j+1}]$, and there is an analytical continuation chain which begins at $(\Delta_0, f_0)$

$$(\Delta_1, f_1), (\Delta_2, f_2), \ldots, (\Delta_N, f_N)$$

such that $\gamma_j \subset \Delta_j$ for all $j = 0, 1, \ldots, N$.

A Theorem due to Riemann on analytical continuation is the Riemann monodromy theorem[10] which simply states

**Theorem 6.17.** *Suppose $\Omega \subset \mathbb{C}$ is a simply connected open set. If an analytic function element, $(\Delta, f)$, can be analytically continued along any path inside $\Omega$, then this analytic function element can be extended to be a single-valued holomorphic function defined on the whole of $\Omega$.*

Now to prove that $C^*$ and $C$ is in fact connected, first note that since $C$ only has at most finitely many singular points and that $C$ and $L_\infty$ intersect in a finite number of points. Hence we can write

$$\overline{C^* \bigcap \mathbb{C}^2} = C.$$

And from point set topology we know that if the set $A$ is connected and

$$A \subset B \subset \overline{A}$$

then the set $B$ is also connected and more importantly $\overline{A}$ is connected.

From this fact and the relation

$$C^* \bigcap \mathbb{C}^2 \subset C^* \subset C = \overline{C^* \bigcap \mathbb{C}^2}$$

we see that it suffices for us to prove that $C^* \bigcap \mathbb{C}^2$ is connected.

Now suppose $C$ is given by the equation $f(x, y)$, by Lemma 6.16 we can write $f(x, y)$ as

$$f(x, y) = y^n + a_1(x) y^{n-1} + \ldots + a_n(x) = 0,$$

[10]The proof of this theorem is found in almost any book in complex analysis

From the discriminant $\mathcal{D}(f)$ of $f$ we can denote the zeros of $\mathcal{D}(f)$ by

$$D = \{x \in \mathbb{C} : \mathcal{D}(f)(x) = 0\}.$$

Let $\pi : C \to \mathbb{C} \equiv \mathbb{C}_x$ denote the projection from the curve $C$ into the $x$-axis. We already know that $\pi^{-1}(D)$ is a finite set and so for $x \in \mathbb{C}\backslash D$, we have just $n$ distinct points

$$(x, y_v(x)) \in C\backslash\pi^{-1}(D),\ (v = 1, \ldots, n)$$

such that

$$f(x, y_v(x)) = 0.$$

Now for the points in $C\backslash\pi^{-1}(D)$ we have that $\frac{\partial f}{\partial y}$, and from the implicit function theorem, each $y_v(x)$ can be regarded as an analytical function element defined on a disc centered at each point.

Let $\Lambda$ be a simple broken line that connects the finite set of points in $D$ and which goes to infinity so that $\Lambda$ cuts the complex plane $\mathbb{C}$ into two disjoint simply connected regions. Denote the upper region by $\Omega$. By the Riemann Monodromy Theorem, we can extend the $n$ function elements to be single valued holomorphic functions defined over the whole of $\Omega$, and we denote these extended functions by $\tilde{y}_v(x)$, $v = 1, \ldots, n$. These extended functions must still satisfy[11]

$$f(x, \tilde{y}_v(x)) = 0.$$

Now we continue $\widetilde{\mu}_v(x)$ along a path $\gamma$ which crosses $\Lambda\backslash D$. The extended functions $\widetilde{\mu}^*_v(x))$ must still satisfy the equation

$$f(x, \widetilde{\mu}^*_v(x)) = 0$$

and must still be one of $\tilde{y}_1(x), \ldots, \tilde{y}_n(x)$. If the original

$$\tilde{y}_\mu(x) \neq \tilde{y}_\nu(x)$$

then after extension we still have

$$\tilde{y}^*_\mu(x) \neq \tilde{y}^*_\nu(x).$$

Or we would get $\tilde{y}_\mu(x) = \tilde{y}_\nu(x)$ after extension along the reversed path $-\gamma$.

If there exists a path $\gamma$ in $\mathbb{C}\backslash D$ such that $\tilde{y}_\mu(x)$ and $\tilde{y}_\nu(x)$ are mutually continuable along $\gamma$, we can define an equivalence relation

$$\tilde{y}_\mu(x) \sim \tilde{y}_\nu(x).$$

Using this equivalence relation to divide $\tilde{y}_1(x), \ldots, \tilde{y}_n(x)$ into equivalence classes $E_1, \ldots, E_l$, one can show that

$$\prod_{\tilde{y}_v(x) \in E_j} (y - \tilde{y}_v(x)) \in \mathbb{C}[x, y]$$

and that

$$f(x, y) = \prod_{j=1}^{l} \prod_{\tilde{y}_v(x) \in E_j} (y - \tilde{y}_v(x)).$$

But if $f(x, y)$ is irreducible, we can only have that $l = 1$, which means that $\tilde{y}_1(x), \ldots, \tilde{y}_n(x)$ all belong to a single equivalence class and so they are all mutually continuable along paths in $\mathbb{C}\backslash D$. Hence, any two points

$$(x_0, \tilde{y}_{mu}(x_0)) \text{ and } (x_1, \tilde{y}_{mu}(x_1))$$

[11] From the identity theorem of analytic functions

in $C\backslash\pi^{-1}(D)$ can be connected by a path. Hence $C\backslash\pi^{-1}(D)$ is connected.

As stated before, the connectedness of $C\backslash\pi^{-1}(D)$ now gives us the connectedness of $C^* = C\backslash S$ and $C$. This gives us the following important theorem.

**Theorem 6.18.** *Suppose $C$ is an irreducible plane algebraic curve. Then $C$ and $C^*$, the set of smooth points of $C$, are both connected sets in $P^2\mathbb{C}$.*

**Corollary 6.19.** *$C^*$ is a Riemann surface (not necessarily compact).*

We can now begin the discussion about the process of normalization. That is given an irreducible curve $C \subset P^2\mathbb{C}$ we can construct a compact Riemann surface $\tilde{C}$ and a holomorphic mapping

$$\sigma : \tilde{C} \to P^2\mathbb{C}$$

such that

$$\sigma(\tilde{C}) = C.$$

**Definition 6.20.** Suppose $C$ is an irreducible plane algebraic curve, and $S$ is the set of its singular points. If there exists a compact Riemann surface $\tilde{C}$ and a holomorphic mapping

$$\sigma : \tilde{C} \to P^2\mathbb{C},$$

such that

(1) $\sigma(\tilde{C}) = C$,
(2) $\sigma_{-1}(S)$ is a finite set,
(3) $\sigma : \tilde{C}\backslash\sigma^{-1}(s) \to C\backslash S$ is injective,

we call $(\tilde{C}, \sigma)$ the normalization of $C$. If there is no danger of confusion we simply say that $\tilde{C}$ is the normalization of $C$.

A question that arises from the definition is whether there are multiple normalizations of one irreducible algebraic curve. As it turns out, normalization is in fact unique up to isomorphism. To show this we will first start of with a lemma needed to prove the uniqueness.

**Lemma 6.21.** *Suppose, $\tilde{C}$ and $\tilde{C}'$ are Riemann surfaces, and*

$$h : \tilde{C} \to \tilde{C}'$$

*is a surjective holomorphic mapping which is injective on a dense open subset of $\tilde{C}$. Then $h$ is a biholomorphic mapping.*

*Proof.* It suffices to show that in the neighborhood of every point $p \in \tilde{C}$, $h$ is locally biholomorphic. By Lemma 2.12 we know that $h$ can locally be represented by

$$w = z^\mu,$$

where $\mu$ is a positive integer. Clearly, if $\mu > 1$, $h$ will not be injective the set $\{z : z \neq 0\}$, and in particular not injective on a dense subset of $\tilde{C}$. Hence, in order for $h$ to be injective we must have that $\mu = 1$, which implies that $h$ is locally biholomorphic in the neighborhood of each point. □

With this tool with us we can now prove the main theorem for uniqueness of normalization.

**Theorem 6.22.** *The normalization of an algebraic curve $C$ is unique up to isomorphism, that is, if $(\tilde{C}, \sigma)$ and $(\tilde{C}', \sigma')$ are normalizations of $C$, then there exists an isomorphism (biholomorphic mapping),*

$$\tau : \tilde{C} \to \tilde{C}'$$

*such that the following diagram commutes:*

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\tau} & \tilde{C}' \\ \sigma \searrow & & \swarrow \sigma' \\ & C & \end{array}$$

*Proof.* If $S$ denotes the singular points of $C$, then the biholomorphic mapping $(\sigma')^{-1} \circ \sigma$ restricted on the sets

$$\tilde{C} \backslash \sigma^{-1}(S) \xrightarrow{\sigma} C \backslash S \xrightarrow{(\sigma')^{-1}} \tilde{C}' \backslash (\sigma')^{-1}(S).$$

can be continuously extended to the whole of $\tilde{C}$. By Lemma 6.21, this extended map is a biholomorphic mapping from $\tilde{C}$ to $\tilde{C}'$ and we denote this map by $\tau$. Then this will make the diagram commute. □

By this we have proved that when a normalization does exist it is unique (up to isomorphism), but given any algebraic curve can we always be sure that such normalization in fact exist? This is a question we cannot answer yet, but with a little more tools at our hands we will be able to answer the question.

An important tool used to prove the existence of normalization is the Weierstrass polynomial. Let $\mathbb{C}\{x\}$ and $\mathbb{C}\{x, y\}$ be rings of holomorphic functions defined in some neighborhood of $0 \in \mathbb{C}$ and $(0, 0) \in \mathbb{C}^2$ respectively, that is

$$\mathbb{C}\{x\} = \left\{ f = \sum_{m=0}^{infinity} a_m x^m \right\}$$

$$\mathbb{C}\{x, y\} = \left\{ f = \sum_{m,n=0}^{infinity} a_{mn} x^m y^n \right\}$$

where each power series can have a different radius of convergence. We denote $\mathbb{C}\{x, y\}$ by $\mathcal{O}$.

**Definition 6.23.** $w \in \mathcal{O}$ is said to be a Weierstrass polynomial with respect to $y$ if

$$w = y^d + a_1(x) y^{d-1} + \ldots + a_d(x),$$
$$a_j(x) \in \mathbb{C}\{x\} \text{ and } a_j(0) = 0 \text{ for all } j = 1, \ldots, d.$$

The main result of Weierstrass polynomials is the Weierstrass preparation theorem, which states that under certain conditions every $f \in \mathcal{O}$ can be uniquely written as $f = u \cdot w$, where $u$ is a unit in $\mathcal{O}$ and $w$ is a Weierstrass polynomial.

**Theorem 6.24.** *(Weierstrass preparation theorem).*

*If $f \in \mathcal{O}$, and $f(0, y)$ is not identically 0, then inside a suitable neighborhood of $(0, 0)$, $f$ has a unique representation*

$$f(x, y) = u(x, y) w(x, y),$$

*where $w(x, y)$ is a Weierstrass polynomial and $u(x, y)$ is a unit of $\mathcal{O}$.*

**Corollary 6.25.** *$\mathcal{O}$ is a U.F.D..*

The consequence of corollary 6.25 is that any $f \in \mathcal{O}$ can be expressed as

$$f = f_1 \cdot f_2 \cdots f_L,$$

where each factor is irreducible in $\mathcal{O}$, and unique up to order and up to units in $\mathcal{O}$.

By combining multiple factors together, then the above factorization can be written as

$$f = f_1^{m_1} \cdots f_l^{m_l}$$

where the irreducible factors $f_1^{m_1}, f_2^{m_2}, \ldots, f_l^{m_l}$ are all distinct.

We will now investigate the local structure of plane algebraic curves, since, as we have seen, the local structure around a singular point may behave differently than the local structure of a smooth point. We start by examining the local structure of any point $p \in C$, and we wish to choose a coordinate system such that $p = (0,0)$ and the affine equation of $C$ satisfies lemma 6.16.

**Lemma 6.26.** *For any plane algebraic curve $C$, with $p \in C$, we can choose a coordinate system such that*

$$p = (0,0) \in \mathbb{C}^2 \subset P^2\mathbb{C}$$

*and such that the affine equation of $C$, $f(x,y) = 0$, satisfies*

$$f(x,y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x),$$

*where $a_j(x) \in \mathbb{C}[x]$ with deg $a_j(x) \leq j$, or $a_j(x) \equiv 0$.*

*Proof.* We can always choose a suitable translation such that $p = (0,0)$, and the rest follows from lemma 6.16. ☐

With lemma 6.26 we can write the affine equation of a plane algebraic curve $C \subset P^2\mathbb{C}$ in the form

$$f(x,y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x) = 0.$$

If $C$ is an irreducible curve, then $f$ is irreducible in $\mathbb{C}[x][y]$, and restricting $f$ to a neighborhood of $p = (0,0)$, we consider $f$ as an element in $\mathbb{C}\{x\}[y]$ and there it is still possible to factor $f$ into a product of irreducible factors

$$f = f_1 \cdot f_2 \cdots f_l,$$

Now, regardless of whether we are in $\mathbb{C}[x][y]$ or $\mathbb{C}\{x\}[y]$ the discriminant will by definition still be the same, and since $f$ is irreducible in $\mathbb{C}[x][y]$, we have that

$$\mathcal{D}(f) \neq 0$$

and so the factorization of $f$ will not contain any multiple factors in $\mathbb{C}\{x\}[y]$ either.

**Definition 6.27.**

(1) Suppose that $f \in \mathcal{O} = \mathbb{C}\{x,y\}$, $f(0,0) = 0$, then

$$V = \{(x,y) \in \mathbb{C}^2 : |x| < \rho, |y| < \varepsilon, f(x,y) = 0\}$$

is called the local analytic curve in the neighborhood of $p = (0,0)$. If $f$ is irreducible in $\mathcal{O}$, then $V$ is called an irreducible local analytic curve.

(2) Suppose $f$ has the following factorization in $\mathcal{O}$

$$f = f_1^{m_1} \cdots f_l^{m_l}$$

where each $f_j$ is irreducible in $\mathcal{O}$. Then we can write

$$V = m_1V_1 + \ldots + m_lV_l,$$

where

$$V_j = \{(x,y) \in \mathbb{C}^2 : |x| < \rho, |y| < \varepsilon, f_j(x,y) = 0\} \ (j = 1, \ldots, l),$$

and each $V_j$ is called an irreducible local curve component of $V$.

So locally we can factorize an irreducible plane algebraic curve into several irreducible analytic curve components which pass through the same point. Our aim is to prove that the underlying topological space of every local analytic curve component corresponds to a disc, so that the local underlying topological space of an algebraic curve in the neighborhood of a point is made up of several discs attached together at their centers. Of course if there is only one irreducible analytic curve component at some given point then the underlying topological space is just a disc. This is the concept of *local normalization* and as we shall see later, we can use this to obtain the normalization of the whole curve.

At this point we will make use of the Weierstrass polynomial and show that the required disc can be found.

**Lemma 6.28.** *Suppose $f$ is an irreducible Weierstrass polynomial,*

$$f(x,y) = y^k + a_1(x)y^{k-1} + \ldots + a_k(x).$$

*Then there exists a disc,*

$$D = \{x \in \mathbb{C} : |x|\rho\},$$

*such that for each fixed $x \neq 0$ in $D$, $f(x,y)$ as a polynomial in $y$ has only simple roots.*

*Proof.* Since $f$ is an irreducible Weierstrass polynomial,

$$\mathcal{D}(f)(x) \not\equiv 0.$$

So $\mathcal{D}(f)(x)$ can only have isolated zeros. And since $f(0,y) = y^k$ has multiple roots, then $\mathcal{D}(f)(0) = 0$. Since 0 is an isolated root of $\mathcal{D}(f)$, there must be a disc

$$D = \{x \in \mathbb{C} : |x|\rho\},$$

such that for every $x \neq 0$ in $D$ we have $\mathcal{D}(f)(x) \neq 0$. So for these values of $x$, $f$ as a polynomial in $y$ has only simple roots. □

If we let $y_v(x)$, $(v = 1,\ldots,k)$ denote the roots of $f(x,y)$ we can write the Weierstrass polynomial as

$$f(x,y) = y^k + a_1(x)y^{k-1} + \ldots + a_k(x) = \prod_{v=1}^{k}(y - y_v(x)).$$

Also, for each $x \neq 0$ in $D$, $\mathcal{D}(f)(x) \neq 0$, so that

$$f_y(x, y_v(x)) \neq 0 \text{ for all } v = 1,\ldots,k.$$

Now, from the implicit function theorem, every $y_v(x)$ is a locally defined holomorphic function. So, by making a cut in the disc, say the positive x-axis, we can analytically continue $y_v(x)$ inside the cut disc. By the Riemann monodromy theorem we now obtain $k$ analytic function elements $y_v(x)$ $(v = 1,\ldots,k)$, which all satisfy

$$f(x, y_v(x)) = 0.$$

Now analytically continue $y_v(x)$ across the cut around $x = 0$. We denote these by $y_v^*(x)$, and they must still satisfy

$$f(x, y_v^*(x)) = 0.$$

and so $y_v^*(x)$ must be one of $y_1(x), y_2(x), \ldots, y_k(x)$ and for each $y_v^*(x)$ there is a unique $y_v(x)$ (i.e. the correspondence is injective). So $y_1(x), y_2(x), \ldots, y_k(x)$ undergo a permutation $\tau$ by the continuation around $x = 0$.

**Lemma 6.29.** *In order for a Weierstrass polynomial $f$ to be irreducible, it is necessary and sufficient for the above described permutation $\tau$ to be injective.*

*Proof.* Suppose the permutation is not injective, so that $\tau$ factors into mutually disjoint cycles

$$(y_{11}(x), \ldots, y_{ls_1}(x)), \ldots, (y_{l1}(x), \ldots, y_{ls_l}(x)).$$

Every such cycle gives rise to a polynomial

$$f_i = \prod_{j=1}^{s_i} (y - y_{ij}(x)) = y^{s_i} + b_1(x) y^{s_i - 1} + \ldots + b_{s_i}(x)$$

where

$$b_1(x) = -\sum_{\lambda=1}^{s_1} y_{i\lambda}(x)$$

$$b_2(x) = -\sum_{1 \le \lambda 1 \le \mu \le s_i} y_{i\lambda}(x) y_{i\mu}(x)$$

.......................................

$$b_{s_i}(x) = (-1)^{s_i} y_{i1}(x) \cdots y_{is_i}(x).$$

The coefficients $b_j(x)$ are invariant under the permutation and so are holomorphic inside $D \backslash 0$, the disc with the origin removed. More, since they are bounded in a neighborhood of the origin, they are holomorphic on the whole of $D$. Thus

$$f_i \in \mathbb{C}\{x\}[y] \ (i = 1, \ldots, l).$$

But since

$$f = f_1 \cdot f_2 \ldots f_l$$

and $f$ is irreducible, we must have that $l = 1$ and $s_1 = k$. Contradicting the fact that the permutation is not injective. □

The following theorem will now tell us the property of the underlying topological surface.

**Theorem 6.30.** *Under the hypothesis of lemma 6.28, define*

$$g : \Delta \to \mathbb{C}^2$$

*by*

$$t \mapsto (t^k, y_v(t^k)),$$

*on the disc*

$$\Delta = \{t \in \mathbb{C} : |t| < \rho\}.$$

*We then have*

(1) *$g$ is a well-defined holomorphic mapping in $\Delta$.*
(2) *$g$ is an injective mapping from $\Delta$ into the local analytical curve*

$$V = \{(x, y) \in \mathbb{C}^2 : |x| < \rho, |y| < \varepsilon, f(x, y) = 0\}$$

*furthermore, $g$ is a biholomorphic map from $\Delta \backslash \{0\}$ onto $V \backslash \{(0, 0)\}$*

*Proof.* (1) When $t$ wraps once around the origin $O \in \Delta$, $t^k$ wraps $k$ times around $O$. By lemma 6.29 as $t$ wraps once around the origin, the value of $y_v(t^k)$ remains unchanged. In this way, $y_v(t^k)$ defines a single valued holomorphic function for $0 < |t| < \rho^{1/k}$. Since $y_v(t^k)$ is bounded in a neighborhood of the origin, it is holomorphic on the whole disc $\Delta$. Hence $g$ is holomorphic on $\Delta$.

(2) If

$$g(t') = (t'^k, y_v(t'^k)) = (t^k, y_v(t^k)) = g(t),$$

then we must have that

$$t' = e^{2\pi i l/k} t,$$

and

$$y_v(e^{2\pi i l} t^k) = y_v(t^k),$$

where $e^{2\pi i l} t^k$ denotes the value of $t^k$ after $t^k$ wraps around the origin $l$ times and by the injectivity stated in lemma 6.29 we get that when $x$ wraps around the origin $km$ times, the value of $y_v(x)$ remain unchanged. Hence we must have

$$l = km$$

for some $m \in \mathbb{Z}$, and so

$$t' = e^{2\pi i l/k} t = e^{2\pi i m} t = t.$$

So $g$ is injective. Also, as $t$ varies inside $\Delta$, $y_v(t^k)$ can assume all the possible values of $y_1(x), y_2(x), \ldots, y_k(x)$ ($|x| < \rho$), by lemma 6.29. Hence $g$ maps $\Delta$ onto $V$.

By the implicit function theorem, $V \backslash \{(0,0)\}$ can be regarded as a Riemann surface with local holomorphic coordinate $x$. The mapping

$$g : \Delta \backslash \{0\} \to V \backslash \{(0,0)\}$$

is holomorphic because its local representation is

$$x = t^k,$$

and so from lemma 6.21, the injective and surjective holomorphic mapping $g$ from $\Delta \backslash \{0\}$ onto $V \backslash \{(0,0)\}$ is a biholomorphic mapping.

□

To complete the discussion about normalization we now have to gather up the pieces we have done so far. We have seen that the set of smooth points $C^*$ of an irreducible algebraic curve $C$ is a Riemann surface. But $C = C^* \bigcup S$ is not necessarily a Riemann surface, since there may be analytically irreducible singular points or more than one irreducible local analytic curve component passing trough a singular point. So what the normalization does is to separate the different curve components at every singular point, uniformize the analytically irreducible points and finally constructs a Riemann surface $\tilde{C}$. The normalization theorem ensures that the conformal class of a Riemann surface does not depend on the singularities.

To show how this works we start of with just one singular point $q$. Suppose there are $m$ irreducible local analytic curve components passing through this point. From theorem 6.30 we know that we get $m$ open discs $\Delta_j$ together with $m$ local normalization mappings $g_j$ such that

$$g_j : \Delta_j \backslash \{0\} \to C^*, \; j = 0, \ldots, m$$

where each is a biholomorphic mapping onto $C^*$. Now we construct the set

$$\tilde{C} = C^* \bigcup_{g_1} \Delta_1 \bigcup_{g_2} \Delta_2 \ldots \bigcup_{g_m} \Delta_m$$

where $C^* \bigcup_{g_1} \Delta_1$ is defined as follows.

Consider the set

$$C^* \bigcup \Delta_1$$

and introduce the relation $\sim$ for $p \in \Delta$ by

$$p \sim g_1(p).$$

This gives an equivalence relation and by the definition

$$C^* \bigcup_{g_1} \Delta_1 = C^* \bigcup \Delta_1 / \sim .$$

Since $g_1 : \Delta_1 \backslash \{0\} \to C^*$ is a biholomorphic mapping onto the image set, we see that $C^* \bigcup_{g_1} \Delta_1$ can (with holomorphic coordinates) be made into a Riemann surface. Doing this successively, we obtain the required Riemann surface

$$\tilde{C} = C^* \bigcup_{g_1} \Delta_1 \bigcup_{g_2} \Delta_2 \ldots \bigcup_{g_m} \Delta_m.$$

When the singular set contains more than one point, $S = \{q_1, \ldots, q_l\}$, we do the above procedure (of local normalization) at each singular point and obtain the Riemann surface

$$\tilde{C} = C^* \bigcup_{g_{11}} \Delta_{11} \bigcup_{g_{12}} \Delta_{12} \ldots \bigcup_{g_{1m_1}} \Delta_{1m_1} \bigcup_{g_{21}} \Delta_{21} \ldots \bigcup_{g_{2m_2}} \Delta_{2m_2} \ldots \bigcup_{g_{l1}} \Delta_{l1} \ldots \bigcup_{g_{lm_l}} \Delta_{lm_l}.$$

Clearly $\tilde{C}$ is compact since it is a union of finite compact sets and the desired mapping $\sigma$ from $\tilde{C}$ into $P^2\mathbb{C}$ can be given by

$$\begin{cases} p, p \in C^*, \\ g_{rs_r}(p), p \in \Delta_{rs_r}, \end{cases}$$
$$\text{for } (1 \leq r \leq l \ 1 \leq s_r \leq m_r).$$

Hence $\sigma : \tilde{C} \to P^2\mathbb{C}$ is a normalization of $C$.

**Example 6.31.** A nice example of how algebraic curves are related to Riemann surfaces is the elliptic curve

$$y^2 = 4x^3 - px - q$$

where $p$ and $q$ are functions of a lattice $\Lambda$ satisfying $p^3 - 27q^2$. This particular elliptic function comes from Weierstrass *pe*-function

$$p(z) = \frac{1}{z} + \sum_{w \in \Lambda}{}' (\frac{1}{(z-w)^2} - \frac{1}{w^2}),$$

where $\sum'$ denotes the sum over all non-zero points of the lattice $\Lambda$.

It can be shown that [12]

$$(p'(z))^2 = 4p(z)^3 - 60 \sum_{w \in \Lambda}{}' w^{-4} p(z) - 140 \sum_{w \in \Lambda}{}' w^{-6}.$$

So the elliptic function os actually a differential equation of the Weierstrass *pe*-function.

[12] See Jones and Singerman [7]

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