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Indices of vector fields

Vector fields in the plane

Definition
A vector field On U with singularities in Z is a continuous mapping

$$V : U - Z \to \Re^2 - \{0\}$$

where Z is a finite set in U.

Definition
The index of V at a point P is defined as the winding number of V restricted to the boundary, C_r, of a disc D_r centred at P and with radius r. Thus

$$Index_P V = W(V|_{C_r}, 0)$$

Lemma 1
a) This definition is independent of choice of r.
b) If P is not a point in Z, then $Index_P V = 0$.

Proof:
a) Suppose r and r' are different radius then $V|_{C_{r'}}$ gives a path $\gamma_{r'} : [0,1] \to \Re^2 - \{p\}$ and $V|_{C_r}$ gives another path $\gamma_r : [0,1] \to \Re^2 - \{p\}$. We can construct a homotopy from γ_r to $\gamma_{r'}$ by the function $H(s,t) = V(P + ((1-s)r + sr')(\cos(2\pi t), \sin(2\pi t)))$. Then since any two homotopic paths have the same winding number, the result follows.
b) Since D_r is the disc of radius r about the point P and $\gamma_r : [0,1] \to \Re^2 - \{p\}$ is the path corresponding to $V|_{C_r}$ and we can extend $V|_{C_r}$ to $V|_{D_r} : D_r \to \Re^2 - \{p\}$, there is a homotpy given by $H(t,s) = V|_{D_r}(P + s(r\cos(2\pi t), r\sin(2\pi t)))$, $s,t \in [0,1]$, from γ_r to the constant path at $V|_{D_r}(P)$. Since the winding number of a constant path is zero so must the winding number of γ_r be and the result follows.

Note that the index of P only depends on an arbitrarily small neighbourhood of P.

Next we will define how a vector field acts on a 1-chain:
If $\gamma = n_1\gamma_1 + \ldots + n_r\gamma_r$ is a 1-chain in an open set U, with γ_i paths, and $F : U \to U'$ is a continuous mapping from U to another open set U', define $F_*\gamma$ to be the 1-chain in U' defined by

$$F_*\gamma = n_1(F \circ \gamma_1) + \ldots + n_r(F \circ \gamma_r)$$

Proposition 1
If γ and δ are closed 1-chains in U with the same winding number around all points not in U, then $F_*\gamma$ and $F_*\delta$ are closed 1-chains in U' with the same winding number around all points not in U'.

With the help of this proposition we can now prove:

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Proposition 2
Let V be a vector field with singularities in U. Suppose γ is a closed 1-chain in U whose support does not meet the singular set Z of V, such that $W(\gamma, P) = 0$ for all P not in U. Then

$$W(V_*\gamma, 0) = \sum_{P \in Z} W(\gamma, P) \cdot Index_P V$$

Proof:
Let $Z = \{P_1, \ldots, P_r\}$ and let D_1, …, D_r be disjoint discs centred at the points in Z, all contained in U. Let γ_i be the standard counterclockwise path around the disc D_i, and let $n_i = W(\gamma_i, P_i)$. Then γ_i and $n_1\gamma_1 + \ldots + n_r\gamma_r$ have the same winding number around every point outside U-Z. By proposition 1it now follows that

$$W(F_*\gamma, 0) = n_1 W(F \circ \gamma_1, 0) + \ldots + n_r W(F \circ \gamma_r, 0)$$

Corollary 3
If $W(V_*\gamma, 0) \neq 0$, then V must have at least one non-vanishing index at a point P with $W(\gamma, P) \neq 0$.

Corollary 4
If U contains a closed disk D, and V has no singularities on the boundary circle C of D. Then

$$W(V|_C, 0) = \sum_{P \in D} Index_P V$$

If $W(V|_C, 0) \neq 0$, V must have a singularity with non-vanishing index inside D.

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Changing coordinates

Let $\varphi : U \to U'$ be a diffeomorphism from one open set in the plane to another. At any point P in U, we have the Jacobian matrix defined as

$$J_{\varphi,P} \begin{bmatrix} \frac{\partial u}{\partial x}(P) & \frac{\partial u}{\partial y}(P) \\ \frac{\partial v}{\partial x}(P) & \frac{\partial v}{\partial y}(P) \end{bmatrix}$$

where $\varphi(x, y) = (u(x, y), v(x, y))$. This gives a linear mapping from vectors in the plane to vectors in the plane. If V is a continuous vector field in U, define the vector field $\varphi_* V$ in U' by

$$(\varphi_* V)(P') = J_{\varphi,P}(V(P))$$

where P is the point in U mapped to P' by φ. If V has singularities in Z then V' will have singularities in $Z' = \varphi(Z)$.

Lemma 5
With V and $\varphi_* V$ as above then for any P in U, $Index_{\varphi(P)}(\varphi_* V) = Index_P V$.

Proof:
WLOG assume that P and P' are the origin, that U is a disc containing the origin and that V is not zero in U-{0}. Let J be the Jacobian of φ at 0. By the homotopy $K : U \times [0,1] \to \Re^2$ such that

$$Q \times t \mapsto \begin{cases} \frac{1}{t}\varphi(t \cdot Q) & 0 < t \le 1 \\ J(Q) & t = 0 \end{cases}$$

We can deduce that $Index_0(\varphi_* V) = Index_0(J_* V)$, which will reduce the problem into a case of linear mapping.
Now $H(Q \times t) = (K_t)_*(V)$ gives a homotopy from $J_* V$ to $\varphi_* V$ and so the problem is further reduced into showing that $Index_0(J_* V) = Index_0 V$ for any linear invertible mapping J.
Now there is a path in the space of invertible matrices from J to either the identity matrix I or to the matrix $I' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. If such a path is given by the formula $t \mapsto J_t$, $a \le t \le b$, then the homotopy $H(Q \times t) = (J_t)_*(V)$ gives a homotopy from $J_* V$ to $I_* V$ or to $I'_* V$. Clearly $I_* V = V$, so all there is to prove is that $Index_0(I'_* V) = Index_0 V$.
If $V(x, y) = (p(x, y), q(x, y))$, then by the definition of I',
$(I'_* V)(x, y) = (p(x,-y), -q(x,-y))$.
So we need only to show that if $F(x, y) = (p(x, y), q(x, y))$ and $R(x, y) = (x, -y)$ the mappings $R \circ F \circ R$ and F, when restricted to a small circle, have the same winding number around the origin.
But by the definition of the winding number we have:

$$W(\gamma, 0) = \frac{1}{2\pi} \sum (v_i(x_i, y_i) - v_i(x_{i-1}, y_{i-1}))$$

where v_i is the corresponding angle function. Now applying R to each v_i from the right and left in this expression we see that it will be indifferent to R. Hence the winding number will be the same.

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Lemma 6
Suppose V and W are continuous vector fields with no singularities on an open neighbourhood U of a point P. Let $D \subset U$ be a closed disc centred at P. Then there is a vector field V' with no singularities in U such that:

i) V' and V agree on U-D
ii) V' and W agree on some neighbourhood of P

Vector fields on a sphere

A vector field V on a sphere S assigns to each point a vector V(P) in the tangent space, T_PS to S at P, where the tangent space is all vectors orthogonal to the vector P. Any such vector field V may have a finite set of singularities Z. Let $\vartheta : S^2 \to \Re^2$ be stereographic projection of the sphere from the north pole.

Lemma 7
The inverse, φ, of the stereographic projection takes points in the plane to

$$\varphi(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right) \text{ in } S^2.$$

The Jacobian matrix $J_{\varphi,P}$ of φ at $P = (x, y)$ maps $\Re^2$ one-to-one onto the tangent space to S^2 at the point $\varphi(P)$. If V is a vector field on S^2, and $\varphi_* V$ is the vector field on $\Re^2$ defined by the equation

$$J_{\varphi,P}((\varphi_* V)(P)) = V(\varphi(P)),$$

then $\varphi_* V$ is continuous at P if V is continuous at $\varphi(P)$.

Now we've reached the main topic in this essay, namely to show that there can't exist a vector field on S^2 with no singularities. I.e. we can't comb the "hairs" on the sphere smoothly with out any bald spots.
Suppose V is a continuous vector field with no singularities on S^2. Then $\varphi_* V$ is a continuous vector field in $\Re^2$ with no singularities. Let C_r be the boundary of a disc centred at the origin in $\Re^2$. By corollary 4, the winding number of $\varphi_* V$ around C_r is zero. Now, in order to show that this is not true we need to obtain a contradiction, i.e. that the winding number is not equal to zero.
For a large r, C_r can be thought of as a small circle around the north pole of S^2. The winding number of $\varphi_* V$ around such a circle is not zero, even though it comes from a vector field that is not zero in the disc near the north pole.
To be a little more strict we give this as a proposition.

Proposition 8
For any vector field with singularities V on S^2,

$$\sum_{P \in Z} Index_P V = 2.$$

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Proof:

There is a vector field on S^2 with $\sum_{P \in Z} Index_P V = 2$. For example, if W is the vector field on $\Re^2$ given by $W(x, y) = (x^2 - y^2, 2xy)$, then $V = \varphi_* W$ is a vector field on the complement of the north pole with one singularity of index 2 at the south pole.
Now we'll show that the sum of the indices of any two such vector fields V and W on S^2 is the same. Let P be a point where neither has a singularity. By lemma 6, replacing V by another vector field V' with the same indices as V, we can assume that V and W agree in some neighbourhood of P. Using stereographic projection from P, one has two vector fields on the plane that agree outside some large disc that contains all the singularities of either vector fields. Taking a larger circle C_r, their winding numbers around C_r will be the same, and with corollary 4 we get that the sum of their indices is the same.