Fundamental groups and coverings

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Fundamental groups and coverings

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Abstract: Some abstract things will be put here eventually...

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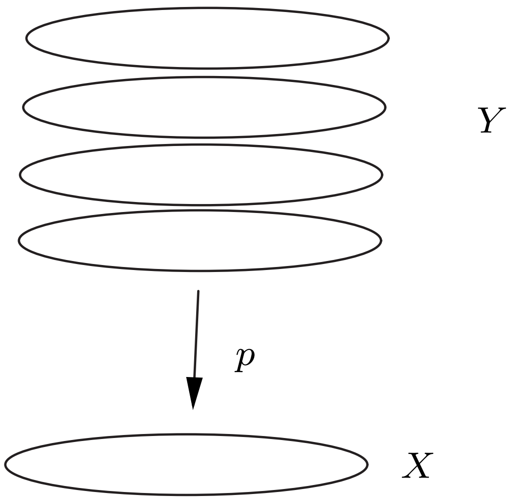
Covering spaces

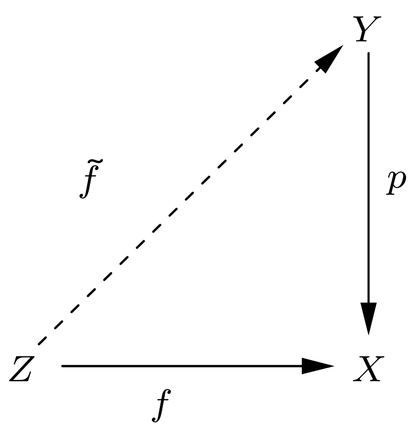
Let X and Y be topological spaces, a **covering space** is a continuous mapping $P : Y \to X$ such that each point of X has an open neighbourhood N such that $p^{-1}(N)$ is a disjoint union of open sets, each which is mapped homeomorphically onto N.

An **isomorphism** between coverings $p : Y \to X$ and $p' : Y' \to X$ is a homeomorphism $\varphi : Y \to Y'$ such that $p' \circ \varphi = p$.

If we can take N to be all of X we say that the covering is **trivial**.



If $|p^{-1}(x)| = n < \infty$ then the covering is called an n-sheeted covering.



If $p : Y \to X$ is a covering and $f : Z \to X$ is a continuous mapping, then a continuous mapping $\tilde{f} : Z \to Y$ such that $p \circ \tilde{f} = f$ is called a **lifting** of f.

Lifts are *unique* up to mapping of one point if the space Z is connected, that is if $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts of f such that $\tilde{f}_1(x) = \tilde{f}_2(x)$ then $\tilde{f}_1 = \tilde{f}_2$.

Paths in X lifts to unique paths in Y. Let $p : Y \to X$ be a covering and let $\gamma : [0,1] \to X$ be a continuous path in X. Let $y \in Y$ such that $p(y) = \gamma(0)$. Then there is a unique continuous path $\tilde{\gamma} : [0,1] \to Y$ such that $\tilde{\gamma}(0) = y$ and $p \circ \tilde{\gamma}(t) = \gamma(t)$.

Theorem 1 (Monodromy Theorem) *Let $p : Y \to X$ be a covering. Let γ_0, γ_1 be paths in X starting and ending at the same points so that γ_0 is homotopic to γ_1. Let $\widetilde{\gamma_0}$ and $\widetilde{\gamma_1}$ be lifts of γ_0 and γ_1 starting at the same point above $\gamma_0(0)$. Then $\widetilde{\gamma_0}$ is homotopic to $\widetilde{\gamma_1}$, and in particular $\widetilde{\gamma_0}(1) = \widetilde{\gamma_1}(1)$*

An **action of a group** G on a space Y (left action) is a mapping $G \times Y \to Y$ given by $(g, y) = g \cdot y$, such that

1. $g \cdot (h \cdot y) = (g \cdot h) \cdot y$ for all $g, h \in G$ and $y \in Y$;

2. $1_d \cdot y = y$ for all $y \in Y$;

3. the mapping $y \mapsto g \cdot y$ is a homeomorphism of Y for all $g \in G$.

Thus G defines a group homeomorphism of Y. Two points y_1 and y_2 in Y are in the same **orbit** if there is an element $g \in G$ such that $g(y_1) = y_2$. This is an equivalence relation and the **orbit space** $X = Y/G$ is the set of equivalence classes.

The **projection map** $p : Y \to X$, maps a point to its orbit and X is equipped with the **quotient topology**.

G is said to act **properly discontinuous** if any point in Y has a neighbourhood V such that $g(V) \cap V = \emptyset$ for all but a finite number of $g \in G$. If this is the case for all $g \in G$, then the action is called **freely and properly discontinuously** or **evenly** for short.

A covering $p : Y \to X$ is called a **G-covering** if it arises from an even action of a group G.

Isomorphisms of G-coverings are isomorphisms of coverings which commutes with the action of G. That is, $p : Y \to X$ and $p' : Y' \to X$ are isomorphic if there is a homeomorphism $\varphi : Y \to Y'$ such that $p' \circ \varphi = p$ and $\varphi(g \cdot y) = g \cdot \varphi(y)$, $g \in G$ and $y \in Y$.

The **trivial** G-covering of X is the product $X \times G \to X$, where G acts by left multiplication on the left factor in $X \times G$.

Lemma 2 *Any G-covering is locally trivial as a G-covering*

Note that if $p : Y \to X$ is a G-covering, then any point in X has a neighbourhood N such that the G-covering $p^{-1}(N) \to N$ is isomorphic to the trivial G-covering $N \times G \to N$.

For any covering $p : Y \to X$ the **group of covering transformations** or **deck transformations** is defined by

$$Aut(Y/X) = \{\varphi : Y \to Y : \varphi \text{ is a homeomorphism and } p \circ \varphi = p\}$$

This is a group by composition of mappings and it acts on Y in the sense of G-coverings. It it called the **automorphism group** of the covering.

If $Y \to X$ is a trivial n-sheeted covering and X is connected, then $Aut(Y/X)$ is isomorphic to the symmetric group S_n.

If the covering is a G-covering, then there is a canonical homomorphism from G to $Aut(Y/X)$ that takes g to the homeomorphism $y \mapsto g \cdot y$.

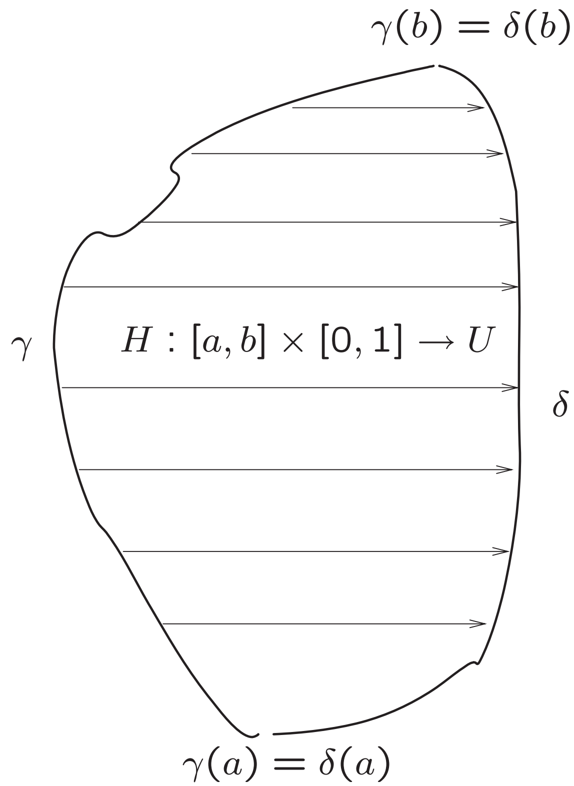
This is an injective homomorphism, but not surjective in general.

Proposition 3 *If $p : Y \to X$ is a G-covering, and Y is connected, then the canonical homomorphism $G \to Aut(Y/X)$ is an isomorphism.*

For a G-covering, G acts transitively on each fiber $p^{-1}(x)$ of p. That is, for each y and y' in a fiber $p^{-1}(x)$ there is an element $g \in G$ such that $g \cdot y = y'$.

Moreover, this action is faithful. That is the element taking y to y' is unique.

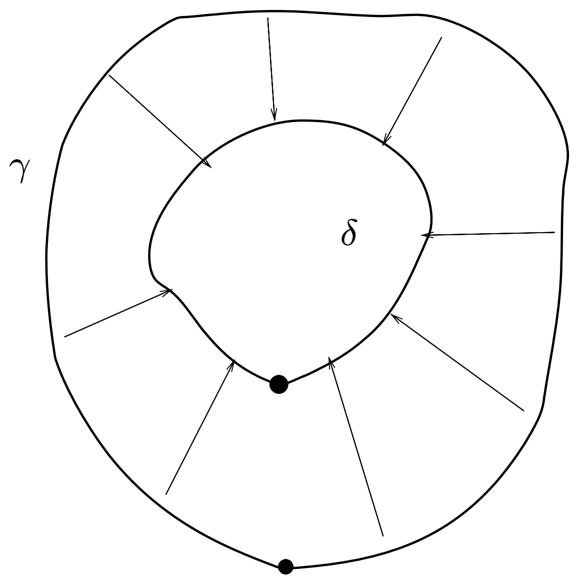
Proposition 4 *Let $p : Y \to X$ be a covering, with Y connected and X locally connected. Then $Aut(Y/X)$ acts evenly on Y. If $Aut(Y/X)$ acts transitively on a fiber of p, then the covering is a G-covering with $G = Aut(Y/X)$*



If $\gamma : [a,b] \to U$ and $\delta : [a,b] \to U$ are two paths with the same start and end points, then a **homotopy from γ to δ with fixed end points** is a continuous mapping $H : [a,b] \times [0,1] \to U$ such that

1. $H(t,0) = \gamma(t)$ and $H(t,1) = \Delta$ for all $a \leq t \leq b$.
2. $H(a,s) = \gamma(a) = \delta(a)$ and $H(b,s) = \gamma(b) = \delta(b)$ for all $0 \leq s \leq 1$.

γ and δ are called **homotopic with fixed end points** if there is such a homotopy.



$H : [a,b] \times [0,1] \to U$

If γ and $\delta : [a,b] \to U$ are closed paths in U, then a **homotopy from γ to δ through closed paths** is a continuous mapping $H : [a,b] \times [0,1] \to U$ such that

1. $H(t,0) = \gamma(t)$ and $H(t,1) = \Delta$ for all $a \le t \le b$.
2. $H(a,s) = H(b,s)$ for all $0 \le s \le 1$.

The paths γ and δ are called **homotopic closed paths** if there is such a homotopy.

Now, the product of paths makes sense by defining it as follows.

If γ is a path from x to x' and δ is a path from x' to x'', then the **product path**, $\gamma \cdot \delta$, is a path from x to x'':

$$\gamma \cdot \delta(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2, \\ \delta(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

If γ is a path from x to x', there is a **inverse path** γ^{-1} from x' to x:

$$\gamma^{-1}(t) = \gamma(1-t), \quad 0 \leq t \leq 1$$

Also, for any point x there is a **constant path** at x:

$$\epsilon_x(t) = x, \quad 0 \leq t \leq 1$$

Now, up to homotopy, these operations satisfy the group axioms and we get the following definition.

Definition 1 **The fundamental group of X with base point x**, *denoted $\pi_1(X, x)$ is the set of equivalence classes of loops at x.*

Write $[\gamma]$ for the class of the loop γ. The identity is the class $e = [\epsilon_x]$ and the product is defined to be $[\gamma] \cdot [\delta] = [\gamma \cdot \delta]$. This product is well-defined and makes $\pi_1(X, x)$ into a group.

If $f : Y \to X$ is a continuous function, and $f(y) = x$, then f determines a homomorphism of groups:

$$f_* : \pi_1(Y, y) \to \pi_1(X, x)$$

This takes paths $[\gamma]$ to $[f \circ \gamma]$.