History of Riemann surfaces

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Figure 1: Bernhard Riemann, (1826-1866)

Abstract

Riemann surfaces have an appealing feature to mathematicians (and hopefully to non-mathematicians as well) in that they appear in a variety of mathematical fields.

The point of the introduction of Riemann surfaces made by Riemann, Klein and Weyl (1851-1913), was that Riemann surfaces can be considered as both a one-dimensional complex manifold and an algebraic curve.

Another possibility is to study Riemann surfaces as two-dimensional real manifolds, as Gauss (1822) had taken on the problem of taking a piece of a smooth oriented surface in Euclidean space and embedding it conformally into the complex plane.

A fourth perspective came from the uniformisation theory of Klein, Poincaré and Koebe (1882-1907), who showed that every Riemann surface (which by

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definition is a connected surface equipped with a complex analytic structure) also admits a Riemann metric.

This is a short survey about the history of Riemann surfaces and the development of such surfaces from Bernard Riemann's doctoral thesis and some of the later results made by poincaré.

Contents

1 Introduction

Riemann surfaces have an appealing feature to mathematicians (and hopefully to non-mathematicians as well) in that they appear in a variety of mathematical fields.

The point of the introduction of Riemann surfaces made by Riemann, Klein and Weyl (1851-1913), was that a Riemann surface can be considered as both a one-dimensional complex manifold and an algebraic curve.

Another possibility is to study Riemann surfaces as two-dimensional real manifolds, as Gauss (1822) had taken on the problem of taking a piece of a smooth oriented surface in Euclidean space and embedding it conformally into the complex plane.

A fourth perspective came from the uniformisation theory of Klein, Poincaré and Koebe (1882-1907), who showed that every Riemann surface (which by definition is a connected surface equipped with a complex analytic structure) also admits a Riemann metric.

In the case where the surface is a closed surface of genus g this means that neighbourhoods on the surface are isomorphic to neighbourhoods on the round sphere, the flat Euclidean plane or the constant negative curvatured hyperbolic plane for $g = 0$, $g = 1$ or $g > 1$ respectively.

Most of the material in this paper is from the book *Mathematics and its History* [9], and so are most of the pictures. The picture of the icosahedral tesselation of the sphere comes from the book *Geometry of Surfaces* [8] and the pictures of Klein's tesselations are from *Non-Euclidean Tesselations and their Groups* [7].

2 The begining

In the seventeenth century the mathematicians were discovering the power series and the first theory of power series began with the publication of the series

$$log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

by Mercator (1668). This was obtained by integrating the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

term by term. The key to finding power series is finding series expansion of simple algebraic functions and once that is done, term by term integration and Newton's method of series inversion[1] yield power series for all the common functions.

The crucial step was perhaps when Newton discovered the general binomial theorem,

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots,$$

which gives the expansions of functions such as $\sqrt{1 - t^2} = (1 - t^2)^{1/2}$.

Even though the generality of the power series $a_0 + a_1x + a_2x^2 + \cdots$, every function $f(x)$ is not expressible as a power series. This is easy to see if one takes a function that tends to infinity as $x \to 0$, since the power series has value a_0 as $x \to 0$.

For other functions, such as $f(x) = x^{1/2}$, the behavior at 0 disallows powerseries for a more subtle reason. These functions have a branching behavior at 0, that is, they are many valued and hence are not functions in a strict sence. Functions such as $x^{1/2}, x^{1/3}, x^{1/4}$ and so on are two-valued, three-valued and four-valued, respectively. The many valued behavior of functions is typical for algebraic functions. An algebraic function y is a function of x if x and y satisfy a polynomial equation $p(x, y) = 0$.

Algebraical functions are not in general expressible by radicals[2] but Newton remarkably discovered (1671) that any algebraic function y can be expressed as a fractional power series in x:

$$y = a_0 + a_1x^{r_1} + a_2x^{r_2} + a_3x^{r_3} + \cdots$$

[1] Used in his work *De methodis* and *De analysis* (1665)

[2] expressed in a finite expression of $+, -, \times, \div$ and fracional powers

where $r_1, r_2, r_3, \ldots,$ are rational numbers. Newton showed by expanding each fractional power in a series that in the neighbourhood of x, the behaviour of y is like that of a finite sum of fractional powers.

The fractional powers themselfes were not properly understood until the variables x and y where taken to be complex. This was done in the nineteenth century by Puiseux (1850) after whom the fractional power series expansion of algebraic functions have been named *Puiseux expansions*.

The problem with multivalued functions still existed however, and at the time of the introduction of complex functions, a 25 year old mathematicians dissertation on the foundations of a general theory of complex functions would initiate a systematic study of topology and also revolutionize algebraic geometry and lay the way for his famous approach to differential geometry.

3 George Friedrich Bernhard Riemann (1826-1866)

George Friedrich Bernhard Riemann was born on the 17th of September 1826 in Breselenz, Hanover (now Germany). Even though he had a short career as a mathmatician, a little more than ten years, his work has been an enormous influence on the modern mathematics.

His father was a Lutherian minister and so when Bernhard enrolled the university of Göttingen in the spring of 1846 he was encouraged to study theology and thus entered the theology faculty. After attending some mathematical lectures he asked his father if he could transfer to the faculty of philosophy so that he could study mathematics. Luckily his father granted this and Bernhard thus became a studen of Moritz Stern and Gauss.

Gaus at the time did however not like lecturing very much and Riemann only attended one course by Gauss in elementary mathematics. There is no evidence that Gauss recognised Riemann's genius, but Stern certainly did realise that he had a remarkable student.

Bernhard moved from Göttingen in 1847 to Berlin to study under Direchlet about potential theory and partial differential equations, number theory and the theory of integration. He also attended lectures on analytical mechanics and higher algebra given by Jacobi and lectures on elliptic functions given by Eisenstein.

In the year 1849 Bernhard returned to Göttingen and attended Wilhelm Weber's lectures on mathematical physics and devoted himself to studies in physics, which seems to have influenced his way of dealing with complex functions.

1851 he gratuated from Göttingen with his famous dissertation *Inauguraldissertation*.

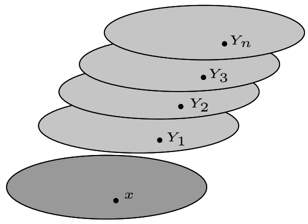
4 Riemann's surfaces

What turned out to be the deepest achievement in his dissertation is the introduction of "surfaces" when dealing with complex functions.

In order to understand the topological form of a complex curve $P(x, y) = 0$ the branch points, that is points α where the Newton-Puiseux expansion of y begins with a fractional power of $(x - \alpha)$.

The nature of such points, as described by Riemann, turned out to be a revolutionary new geometric theory of complex functions.

Riemann's idea was to represent a relation $P(x, y) = 0$ between x and y (complex variables) by covering a plane (or a sphere), representing the x variable, by a surface representing the y variable. Thus, the points on the y surface over a given point $x = \alpha$ were those values of y that satisfy the relation $P(x, y) = 0$. Locally this would look like the picture on the right, where $y_1, y_2, \ldots, y_n$ are the roots of the equation $P(\alpha, y) = 0$.



The sheets above x

If the equation $P(\alpha, y) = 0$ is of degree n in y then there will in general be n sheets of the surface lying above $x = \alpha$. At finitely many values of x, these sheets merge due to coincidence of roots, and the Newton-Puiseux theory says that at such points y behaves lika a fractional power of x at 0.

As an example consider the curve $y^2 = x$ with solutions $y = \pm\sqrt{x}$ above x in the unit disk of the x-plane.

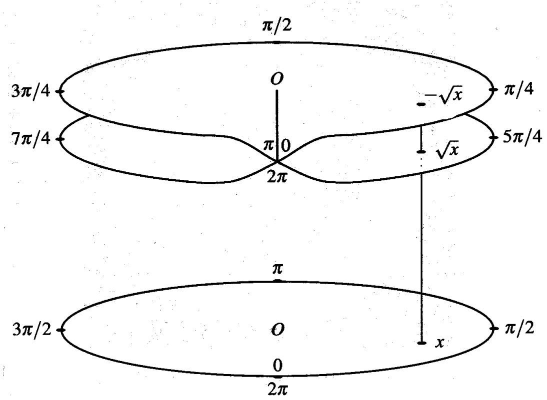


Figure 2: The curve $y^2 = x$.

The values above x are pinched toghether in a region of 0. However, the line of self intersection is only a consequence of representing the relation $y^2 = x$ in fewer dimensions than the four it really requires.

5 The topology of complex projective curves

In order to understand the complete structure of the complex projective curve definded by $y^2 = x$ it is also nesseccary to study the behaviour at ∞, since this is also a branch point similar to the one at 0. In order to see this, substitute x by $1/u$ and y by $1/v$ and notice that we are investigating $v^2 = u$ near $u = 0$ (that is x near ∞).

The topological nature of the relation between x and y can be modeled as a sphere covered by two spheres. This is done by sliting along an arbitrary curve from 0 to ∞ and then cross-joining it again.

This covering express the covering projection map $(x, y) \mapsto x$ from a general point of the curve $y^2 = x$ to its x coordinate and shows that it is two-to-one exept at the branching points 0 and ∞.

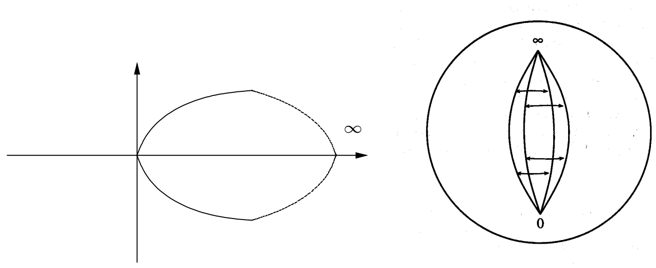


Figure 3: The projective curve $y^2 = x$ and the corresponding slit in the sphere

The best way to observe the structure of the two-sheeted surface is to separate the two sheets of the sphere from eachother and then joining them by the required edges.

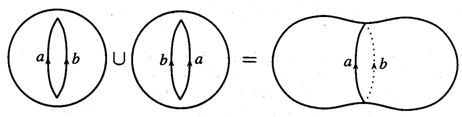


Figure 4: The two sheets joined together

Thus, after identifying edges (“glueing”) the resulting surface is topologically a sphere.

This method extends to all algebraic curves and the Newton-Puiseux theory implies that any algebraic relation $P(x, y)$ can be modeled by a finite-sheeted covering of the sphere, with finitely many branch points.

For a more general curve, consider the cubic curve

$$y^2 = x(x - \alpha)(x - \beta)$$

This is aslo a 2-sheeted covering of the sphere but with branch points 0, α, β and ∞. Thus, if we again slit the sheets from 0 to α and from β to ∞ and then make the proper identifications we obtain, as Riemann also did, a surface topologically equal to a torus.

This discovery proved to be a revelation for the understanding of cubic curves and elliptic functions.

It is not difficult to see that relations of the form

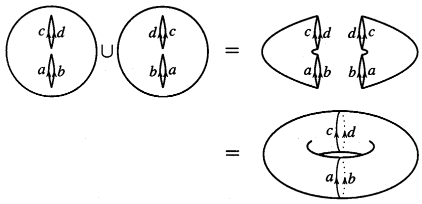


Figure 5: Glueing the torus

$$y^2 = x(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_{2n})$$

will yield Riemann surfaces of genus n, that is, surfaces of the form given in the below figure

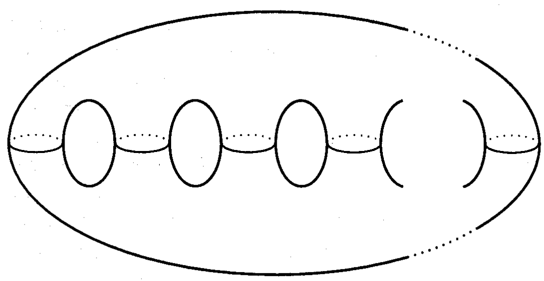


Figure 6: Riemann surface of genus n

The topological importance of genus was established by Möebius (1863) when he showed that any closed surface in ordinary space is toplogically equivalent to a sphere with handles on it.

6 Conformal mappings

The study of conformal mappings is a subject that has been studied since ancient times. The mapping of a sphere onto a plane is a practical problem, when drawing maps of the earth for example.

This has been studied by, among others, Ptolemy (around 150 A.D. the stereographic projection) and G. Mercator (1569, the Mercator projection). Both these projections where conformal, that is angle preserving mappings.

Advances in the theory of conformal mapping were made by Lambert (1772), Euler (1777) and Lagrange (1779), where Lagrange's result was the most general. He combined a pair of differential equations in two real variables into a single equation in one complex variable and arrived at the result that any two conformal maps of a surface of revolution onto the (x, y)-plane where related via a complex function $f(x + iy)$ mapping the plane onto itself. These results were later refined by Gauss (1822), who generalized Lagrange's theorem into conformal maps of an arbitrary surface into the plane.

Conversely, a complex function $f(z)$ defines a map of the z plane onto itself and this map is conformal. This is infact a consequence of the differentiability of f.

Now, Riemann was the first to take the conformal mapping property as a basis of the theory of complex functions and his most famous result in this direction is the Riemann mapping theorem:

Any region of the complex plane bounded by a simple closed curve can be mapped onto the unit disc conformally, and hence by a complex function.

The proof was based partly on appeal to physical intuition using the so called *Dirichlet's principle* and went against the growing tendency toward rigorous proofs in analysis. Stricter proofs were given by Schwartz (1870) and Neumann (1870) but Riemanns faith in the physical roots of complex functions was justified when Hilbert (1900) managed to show the validity of Dirichlet's principle.

7 Elliptic functions

In order to understand elliptic integrals such as

$$\int_0^z \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}}$$

one step is to use the complex integration provided by Cauchy's theorem. Another important step is using the idea of a Riemann surface to visualize the possible paths of integration from 0 to z. The function $1/\sqrt{t(t-\alpha)(t-\beta)}$ is obviously two-valued and, has we have seen, it is represented by a two-sheeted covering of the t sphere, with branch points at $0, \alpha, \beta$ and ∞. Thus the path of integration are curves on this surface, which (as we also have seen previously) topologically is a torus.

Now, the torus contains types of curves that does not bound a piece of the surface. That is curves such as $\mathcal{C}_1$ and $\mathcal{C}_2$ as shown in figure 7.

Since there is no region that is bounded by $\mathcal{C}_1$ and $\mathcal{C}_2$ Green's theorem does not apply, and so the values of the integrals

$$\omega_1 = \int_{\mathcal{C}_1} \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}}$$

$$\omega_2 = \int_{\mathcal{C}_2} \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}}$$

are non-zero. Consequently the integral

$$\Phi^{-1}(z) = \int_0^z \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}}$$

will give different values for each z. That is, for each value $\Phi^{-1}(z) = w$ obtained for a certain path $\mathcal{C}$ from 0 to z we also obtain the values $w + m\omega_1 + n\omega_2$ by adding a detour to $\mathcal{C}$ that winds m times around $\mathcal{C}_1$ and n times around $\mathcal{C}_2$.

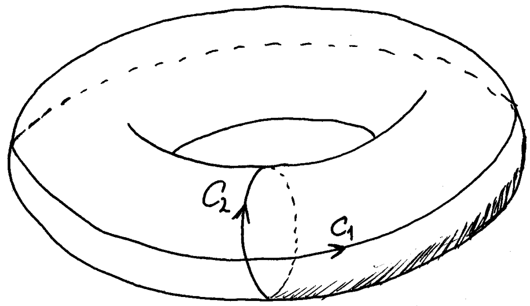


Figure 7: Nonbounding curves of the torus

Hence it follows that the inverse relation $\Phi(w) = z$, the elliptic function corresponding to the integral, satisfies

$$\Phi(w) = \Phi(w + m\omega_1 + n\omega_2)$$

for any integers m, n. That is, Φ is doubly periodic, with periods ω_1 and ω_2. This intuitive explanation of doubly periodicy is due to Riemann who later developed the theory of elliptic functions from this standpoint.

Eisenstein (in 1847) discovered remarkable series expansions of elliptic functions that exhibit the double periodicity and he showed that they could be obtained by expressions such as

$$\sum_{m,n=-\infty}^{\infty} \frac{1}{(z + m\omega_1 + n\omega_2)^2}.$$

Such functions are periodic with periods ω_1 and ω_2 and is in fact identical, up to a constant, with the Weierstrass $\wp$-function. The precise definition of this function is

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}^{\infty} \Big(\frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \Big)$$

and as we shall see, this function plays an important roll in the parametrization of cubic curves.

8 Elliptic curves

Cubic curves of the form

$$y^2 = ax^3 + bx^2 + cx + d \tag{8.1}$$

play an important roll in mathematics and arise in several different areas, such as number theory and in the theory of elliptic functions. Thus one of the great achievements of the nineteenth-century mathematicians was the classification of these curves.

This was first done by Jacobi (1834) but it was not until Riemann (1851) and Poincaré (1901) developed the complex analysis, that this result became more into focus.

Jacobi discovered that a cubic curve such as (8.1) could be parametrized as

$$x = f(z), \qquad y = f'(z)$$

where $f(z)$ and its derivative $f'(z)$ are elliptic functions.

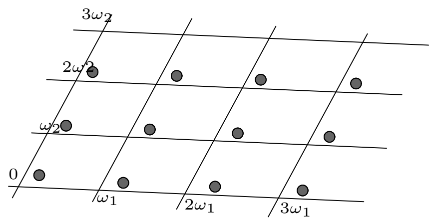
This gave, since f and f' were doubly periodic with periods ω_1 and ω_2, a map of the z plane $\mathbb{C}$ onto the curve (8.1) for which a point on the curve is a preimage of the set of points in $\mathbb{C}$ of the form

$$z + \Lambda = \{z + m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$$

where Λ is the lattis of periods of f generated by ω_1 and ω_2.

Thus, there is a one-to-one correspondence between the points $(f(z), f'(z))$ of the curve and the equivalence classes $z+\Lambda$. Today this is expressed as saying that the curve is *isomorphic to the quotient space* $\mathbb{C}/\Lambda$ of equivalence classes.

It is possible that Jacobi saw that this space is a torus, but it was probably of no interest to him.



The lattice points of Λ

Now the torus is constructed by identifying opposite edges of one of the polygons in the lattice.

Another way of demonstrating the double periodicity of elliptic functions and the parametrization of elliptic curves was done by Weierstrass (1863) when he found his $\wp$-function.

From the doubly periodic function

$$\sum_{m,n=-\infty}^{\infty} \frac{1}{(z + m\omega_1 + n\omega_2)^2},$$

explained earlier, he defined the function

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}^{\infty} \left(\frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right)$$

which is also doubly periodic but with better convergence properties. He showed that with simple calculations with series the function satisfies the relation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where g_2 and g_3 are dependent on ω_1 and ω_2, defined by

$$g_2 = 60 \sum \frac{1}{(m\omega_1 + n\omega_2)^4}, \qquad g_3 = 140 \sum \frac{1}{(m\omega_1 + n\omega_2)^6}.$$

Hence it follows that the point $(\wp(z), \wp'(z))$ lies on the curve

$$y^2 = 4x^3 - g_2 x - g_3$$

and in fact, the above curve is isomorphic to $\mathbb{C}/\Lambda$ where Λ is the lattice of periods of $\wp$. Now the parametrization of all cubic curves (8.1) follows by making a linear transformation.

Once the curve is parametrized using elliptic functions

$$x = f(z), \qquad y = f'(z)$$

addition of points on the curve induce addition of their parameter values. Since the parameters are doubly periodic, this addition becomes addition in $\mathbb{C}$ modulo Λ. In particular, it is now clear that the addition of such points of the cubic curve must have the properties of normal addition, such as commutativity and associativity.

Another reason to use $\mathbb{C}/\Lambda$ as a view of the cubic curve is that it clarifies the question of classification by projective equivalence of cubic curves.

Newton had reduced the cubic curves into curves having a cusp, curves having a double point and three non-singular types, using real projective transformations.

In fact, cubics with a cusp are all equivalent to $y^2 = x^3$ and cubics with a double point are all equivalent to $y^2 = x^2(x+1)$. The three non-singular types does not have any similar equivanlences, however they are all equivalent to the tori $\mathbb{C}/\Lambda$. Thus, the problem was reduced to decide projective equivalence among the non-singular curves.

In 1851, Salomon showed that this was determined by a complex number τ, which was defined geometrically, so the projecive invariance was obvious, and without using elliptic functions.

As it turns out, τ is nothing but the quotient ω_1/ω_2, which means that two non-singular cubic curves are projectively equivalent if and only if their period lattices Λ have the same shape.

9 The uniformization problem

What characterise non-singular cubics so that they can be parameterized by elliptic functions is their topological form. The two lattice periods corresponds to the two different nonbounding circles on the torus.

A representation of the x and y values on a curve by functions of a single parameter z is sometimes called uniform representation and so the problem of parameterizing all algebraic curves in this way came to be known as the unifomization problem.

As with the elliptic case, it became clear that the solution of the uniformization problem of algebraic curves would depend of a better understanding of surfaces, such as to study their topology, the periodicities associated with their closed curves, and the way these periodicities could be reflected in $\mathbb{C}$. These problems were attacked by Poincaré and Klein in the 1880's and their work led to the eventual positive solution of the uniformization problem by Poincaré and Kobe (1907).

What was more important than the solution to the unifomization problem was the preliminary work done by Poincaré and Klein. They discovered that multiple periodicities were reflected in $\mathbb{C}$ by groups of transformations, and that these transformations where of the types

$$z \mapsto \frac{az+b}{cz+d}.$$

Such maps are called linear fractional transformations and they generalize the linear transformations

$$z \mapsto z + \omega_1 \qquad z \mapsto z + \omega_2$$

which we saw where associated to the periods of the elliptic functions.

The transformations $z \mapsto z + \omega_1$ and $z \mapsto z + \omega_2$ are quite simple since they commute and they generate the general transformation $z \mapsto z + m\omega_1 + n\omega_2$ which is a translation of the plane. The more general linear fractional transformation is more complicated and not as easily understood. They do not normally commute and one need to use both algebraic, geometric and topological aspects to undestand them.

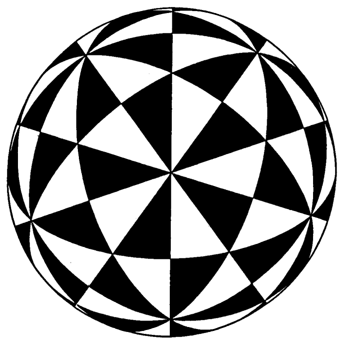
Poincaré (1882) discovered that the linear fractional transformations give a natural interpretation of non-euclidean geometry.

10 Fitting the pieces together

One of the characteristics of the Euclidean plane is the existence of regular tesselations, that is tilings of the plane by regular polygons. Such tilings can be for example the unit square, triangles or as we also have seen lattice polygons. Associated with each tiling is a *group of rigid motions* of the plane that maps the tilings onto itself. As a fact, any motion of the Euclidean plane can be composed from translations $z \mapsto z + a$ and rotations $z \mapsto ze^{i\theta}$.

The sphere admits a finite number of regular tesselations, all obtained by central projections of the regular polyhedra (tetrahedron, cube, octahedron, dodecahedron and icosahedron). The motions that map such a tesselation onto it self can also be expressed as complex transformations by interpreting the sphere as $\mathbb{C} \cup \{\infty\}$, also known as the Riemann sphere, via stereographic projection onto the complex plane. Gauss (1819) found that any motion of the sphere can be expressed by a transformation of the form

$$z \mapsto \frac{az+b}{-\bar{b}z+\bar{a}}$$



An icosahedral tesselation of the sphere

where $a, b \in \mathbb{C}$.

The conformal models of the hyperbolic plane can also be regarded as parts of the complex plane. One model is the unit disk $\{z : |z| < 1\}$ and another model is the upper half-plane $\{z : Im(z) > 0\}$. Their rigid motions, conformal transformations of the complex plane, are complex functions and Poincaré (1882) made the beautiful discovery that they are of the form

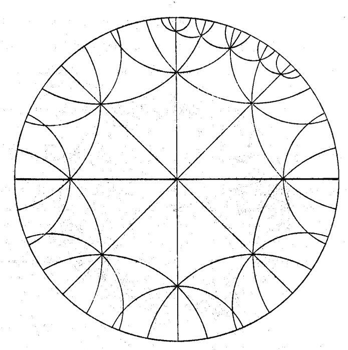
$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$$

for the disk, and

$$z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ for the half-plane. There are infinitely many tesselations of these models since the interior angles of an regular n-gon can be made arbitrary small by increasing its area.

Some of these tesselations where already known before Poincaré gave his complex interpretation of the hyperbolic geometry. For example, a tesselation of equilateral triangles of angle $\pi/4$ was found in an unpublished and (unfortunately) undated work of Gauss.



The Gauss tesselation

Another well known tesselation found by Klein (1879) when he made the astonishing discovery that there is a regular 14-gon with corner angles $2\pi/7$ formed by 336 copies of a triangle with angles $\pi/2$, $\pi/3$ and $\pi/7$. If the sides in this polygon are identified as with the tori example, then the resulting surface S is a genus 3 surface with two vertices, each corresponding to the sets of even and odd verticies of the 14-gon and hence have angle sum 2π.

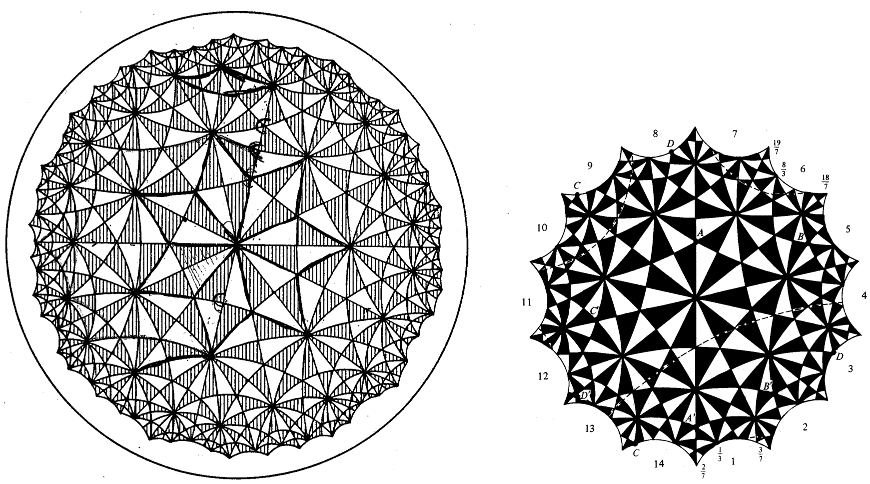


Figure 8: Klein's tesselation of the unit disk

Now the tesselation of the 14-gon can be lifted to the unit disk by lifting the tesselation of the surface S to its universal cover.

In a subsequent paper, Poincaré (1883) explained the geometric nature of linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d}.$$

He showed that each linear transformation of the plane $\mathbb{C}$ is induced by hyperbolic motion of the three-dimensional half-space with boundary plane $\mathbb{C}$.

11 Development of groups and geometry

It is not difficult to realize that group theory is well connected with geometry when studying for example planar motions, however it is not always clear as to what the product of motions would become.

What was a breakthrough in the introduction of group theory into geometry, was the extension of the idea of motion to the whole plane by Möbius (1827), which gave a meaning to the product of motions. Möbius considered all continuous transformations of the plane that preserve straightness of lines and separated several subclasses of these transformations: those that preserve length (congurences), shape (similarities) and parallelism (affinities). He showed that the most general continuous transformations that preserve straightness were just the projective transformations. Thus, Möbius defined in one stroke the notions of congruence, similarity, affinity and projective equivalence as properties that were invariant under certain classes of transformations of the plane.

The classes in question was clearly groups, once one recognized the concept of group. However, it was not until Klein (1872) restated Möbius results in terms of groups that the concept of group was recognized.

Klein's formulation became known as the *Erlanger Programm*[3] and his idea was to associate each geometry with a group of transformation that preserve its characteristic properties.

When geometry is reformulated this way, certain geometrical questions become questions about groups. For example, a regular tesselation corresponds to a subgroup of the full group of motions, consisting of those motions that map the tesselation onto itself. In the case of hyperbolic geometry, the interplay between geometric and group-theoretic ideas was found to be very fruitful. Much of the work of Poincaré and Klein is built on these new geometric, topological and combinatorical ideas.

12 Classification of surfaces

Between the years 1850 and 1880 different lines of research led to the demand for a topological classification of surfaces. There was the classification of polyhedra, descending from Euler and there was the Riemann surface representation of algebraic curves, comming from Riemann (1851, 1857). Also, there was the problem of classifying symmetry groups of tesselations, considered by Poincaré (1882) and Klein (1882).

[3]Simply because it was announced at the University of Erlangen

Finally there was the problem of classifying smooth closed surfaces in ordinary space, done by Möbius (1863).

These four different lines of research became one when it was realized that in each case the surface could be subdivided into faces by edges so that it became a generalized polyhedron. The generalized polyhedra are *closed* surfaces and are described by topologists as *compact* and *bounded*.

Now, the subdivision argument for the invariance of the Euler characteristic[4] $\chi = V - E + F$ applies for any such polyhedron, no matter if its edges are not straight or the faces are not flat. Riemann and Jordan (1866), among many other mathematicians, came to the conclusion that any closed surface is determined, up to homeomorphism, by its Euler characteristic and the different possible Euler characteristics where represented by "normal form" surfaces, being a sphere with a number of handles attatched to it.

Figure 9: General compact orientable surfaces

Now, with the assumption that the surfaces were supposed to be Riemann surfacesa and smoothly imbedded in $\mathbb{R}^3$ it was impossible to yield a purely topological proof. There was also a hidden assumption, namely that of orientability of the surfaces. Thus, a rigorous proof was given later by Dehn and Heegaard (1907).

The closed oriented surfaces turned out to be those described above and in addition there are the non-orientable surfaces, which are not homeomorphic to orientable surfaces.

A non-orientable surface may be defined as a surface containing a *Möbius band*, a non-closed surface discovered independently by Möebius and Listing in 1858. The closed non-orientable surfaces can not occur as Riemann surfaces, nor can they lie in $\mathbb{R}^3$ without self-intersection. The non-orientable surfaces constitute of some of the most important surfaces though, such as the projective plane and they are also determined up to homeomorphism by the Euler characteristic.

The Möbius forms of closed orientable surfaces were given standard polyhedral structures by Klein (1882). These are minimal subdivisions with just one face and, exept for the sphere, with just one vertex. When the Klein subdivision of a surface is cut along its edges, one obtains a fundamental polygon, from which the surface may be reconstructed by identifying the labeled edges. This is the reason why this "inverse" process of building a surface from a polygon is sometimes called *glueing* or *edge glueing* (figure 12).

The standard polygon for the genus g surface has a boundary path of the form

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1},$$

[4] V is the number of vertices, E is the number of edges and F is the number of faces of the polyhedron.

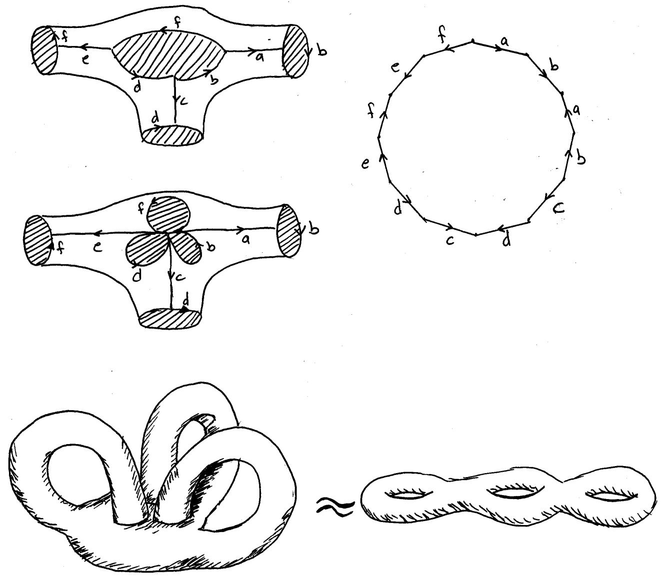


Figure 10: Example of edge glueing

where successive letters denote successive edges and edges with negative exponents have opposite direction. Edges with the same letter are pasted together, with their arrows matching.

Usually, it is more convenient to work with these polygons instead of the surface or its polyhedral structure. The fundamental polygon also gives a very easy calculation of the Euler characteristic χ and shows that it is related to the *genus*[5] g with the formula

$$\chi = 2 - 2g$$

13 Covering surfaces and the universal plane

We have seen that an elliptic funcion defines a mapping of a plane onto a torus. Such mappings are interesting in the topological contex, where they are called universal coverings. In general, a mapping $f : X \mapsto Y$ of a surface X onto a surface Y is called a covering if it locally is a homeomorphism. A covering is called *universal* if the covering surface is a sphere or a plane.

An interesting example of a covering is the mapping of the sphere onto the projecive plane given by Klein (1874). This map sends antipodal points of the sphere to the same point in the projective plane.

Since the sphere can only be covered by itself, the first interesting examples of coverings are those of orientable surfaces of genus ≥ 1. All of these surfaces can

[5]Basically, the genus is the number of "holes" in the surface

be covered by planes, and each non-orientable surface can be doubly covered by an orientable surface in the same way the projective plane is covered by the sphere. Thus, the main thing is to understand the universal covering of orientable surfaces of genus ≥ 1.

The basic idea is due to Schwartz (it became generally known from a letter from Klein to Poincaré). To construct the universal covering of a surface S one takes infinitely many copies of a fundamental polygon F for S and arranges them in the plane so that adjacent copies of F meet in the same way that F meets itself on S.

Thus, for the torus, we get an infinite tiling of the plane with adjacent copies of the fundamental polygon. Since these squares can be realized as squares in the plane, we can impose a Euclidean metric on the torus by defining distance between points[6] on the torus to be the distance between appropriate preimage points in the plane. In particular, the geodesics of the torus is the image of straight lines in the plane.

Clearly, the geometry of the torus is not the same as the geometry of the plane, since there are closed geodesics. But locally, it will be Euclidean as for example, the angle sum of each triangle on the torus is π.

For surfaces of genus > 1, the Gauss-Bonet formula predicts negative curvature and hence the natural covering plane is hyperbolic. The tesselations for general genus > 1 can be realized in a similar way in the hyperbolic plane and were among the hyperbolic tesselations considered by Poincaré (1882) and Klein (1882).

14 Later History

The theory of surfaces and Riemann surfaces now starts to become accepted and with the introduction of the fundamental group by Poincaré, the basics for the foundation of the theory is done. From here a lot of results have been achieved by works of Dehn (1912), Nielsen (1927) and Seifert and Threfall (1934).

The arsenal of techniques and concepts built up by Poincaré in his papers between 1892 and 1904 managed to keep topologists busy for the next 30 years. It was not until Hurewicz discovered higher dimensional analogues of the fundamental group in 1933 that a significant new idea was added to the theory.

[6] Since things are locally, we need the points to be sufficiently close to eachother

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