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Cyclic Trigonal Riemann Surfaces of Genus 4.

Daniel Ying

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Daniel Ying

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Contents

Introduction

Riemann surfaces were first introduced by Riemann in his doctoral dissertation *Foundations for a general theory of functions of a complex variable* in 1851. The use of the Riemann surfaces was as topological aid to the understanding of many-valued functions. Since then the results have been improved, amongst others by F. Klein, but it took until 1913 for the first abstract definition of a Riemann surface to appear in H. Weyl's book *The concept of a Riemann surface.*

A Riemann surface X is a Hausdorff conneced topological space, together with a family $\{(\phi_j, U_j) : j \in J\}$ where U_j is an open cover of X and each ϕ_j is an homeomorphism of U_j onto an open subset of the complex plane. A closed Riemann surface X which can be realized as a 3-sheeted (branched) covering of the Riemann sphere is said to be *trigonal*, and such a covering is called a *trigonal morphism.* A morphism is a branched covering.

To study Riemann surfaces is equivalent to study algebraic curves, thus a trigonal Riemann surface X is represented by an algebraic curve of the form

$$y^3 + yb(x) + c(x) = 0$$

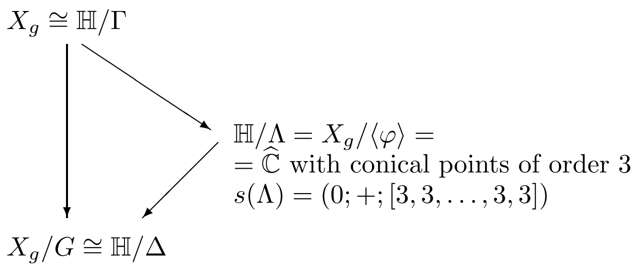
If $b(x) \equiv 0$ then the trigonal morphism is a *cyclic regular covering* and the Riemann surface is called *cyclic trigonal.* A non-cyclic trigonal Riemann surface is said to be a *generic trigonal Riemann surface.*

If X_g is a cyclic trigonal Riemann surface there is an automorphism, φ, of order 3 such that $X_g/\langle\varphi\rangle$ is the Riemann sphere with conic points of order 3, and in fact there are $g+2$ such conical points. φ will be called a trigonal morphism. Cyclic Riemann surfaces have equations

$$y^3 = c(x)$$

A trigonal Riemann surface X_g, of genus g, can be uniformized by a Fuchsian group Γ, that is, $X_g = \mathbb{H}/\Gamma$ and the trigonal morphism gives $g+2$ conical points on the Riemann sphere so the quotient surface is uniformized by a Fuchsian group with signature $s(\Lambda) = (0; +; [3, 3, \ldots, 3, 3])$.

Let $G = Aut(X_g)$ then the quotient surface X_g/G is also uniformized by a Fuchsian group Δ such that $X_g/G = \mathbb{H}/\Delta$.



Accola showed in his paper [1] that the trigonal morphism, φ, is unique for Riemann surfaces of genus $g \geq 5$, however for genus less than 5 the morphism need not be unique as we shall see.
The task of classifying the possible trigonal Riemann surfaces of genus 4 is done by finding the signature groups of Δ and defining suitable epimorphisms from the groups Δ onto groups with order a multiple of 3.
The main result is given by the two theorems:

Theorem 1. *(i) There is a uniparametric family of cyclic trigonal Riemann surfaces $X_4(\lambda)$ of genus 4 with non-normal trigonal morphisms. $Aut(X_4(\lambda)) = D_3 \times D_3$ and $X_4(\lambda)/Aut(X_4(\lambda))$ are uniformized by the Fuchsian group Δ with signature $s(\Delta) = (0; [2, 2, 2, 3])$.*

(ii) There is one cyclic trigonal Riemann surface Y_4 of genus 4 with non-normal trigonal morphisms. $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$ and $Y_4/Aut(Y_4))$ is the sphere with 3 conic points of order 2, 4 and 6 respectively.

Theorem 2. *The space $\mathcal{M}_4^3$ of cyclic trigonal Riemann surfaces of genus 4 form a disconnected subspace of the moduli space $\mathcal{M}_4$ of dimension 3.*

1. *The subspace of $\mathcal{M}_4^3$ formed by Riemann surfaces of genus 4 with automorphism group of order 6 has dimension 2 in $\mathcal{M}_4^3$. The automorphism group of the Riemann surfaces is either C_6 or D_3.*

2. *The subspace of $\mathcal{M}_4^3$ formed by Riemann surfaces of genus 4 with automorphism group of order 12 has dimension 1 in $\mathcal{M}_4^3$. The automorphism group of the Riemann surfaces is either $C_2 \times C_6$ or D_6.*

3. *The subspace of $\mathcal{M}_4^3$ formed by Riemann surfaces $X_4(\Delta)$ of genus 4 with automorphism group of order 36 has dimension 1 in $\mathcal{M}_4^3$. The automorphism group of the Riemann surfaces is $D_3 \times D_3$ and the surfaces admit non-normal trigonal morphisms.*

4. *There are exactly 2 cyclic trigonal Riemann surfaces X_4 and Y_4 of genus 4 with automorphism groups of order 72.*

   (i) X_4 has a normal trigonal morphism and $Aut(X_4) = S_4 \times C_3$.

   (ii) Y_4 has non-normal trigonal morphisms and $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$

Chapter 1

Preliminaries

1.1 Hyperbolic geometry

Hyperbolic geometry takes place in the hyperbolic plane and the hyperbolic geometry can be described in terms of the usual Euclidean geometry by using different models. The two most common models are the upper complex half-plane, denoted $\mathbb{H} = \{x + iy : y > 0\}$, and the unit disc, $\Delta = \{z \in \mathbb{C} : z < 1\}$
The metrics ρ on each model is given by

$$ds = \frac{|dz|}{Im(z)},$$

for the upper half-plane and

$$ds = \frac{2|dz|}{1 - |z|^2},$$

for the unit disc.
These two models will be refered to as the Poincaré models, and the benefits of using these two models are that we can easily describe the circle of points at infinity in each of them, namely $\mathbb{R} \cup \infty$ in the upper half-plane and $\{z \in \mathbb{C} : |z| = 1\}$ in the unit disc.
The group of orientation preserving isometries (maps leaving ρ invariant) of the hyperbolic plane is the extended group of Möbius transformations. The elements are expressed by

$$z \mapsto \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, and the orientation reversing transformations

$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = -1$, for the upper half-plane. The orientation preserving transformations are expressed by

$$z \mapsto \frac{az + \bar{c}}{cz + \bar{a}},$$

where $|a|^2 - |c|^2 = 1$, for the unit disc.
There will be no distinction between the two models, we will denote both models $\mathbb{H}$.
We define a hyperbolic line (h-line) to be the intersection of the hyperbolic plane with a Euclidean circle or straight line which is orthogonal to the circle at infinity. Using this definition it is a well know fact that the following hold:

1. The reflection in an h-line is a ρ-isometry.
2. Any transformation is the product of at most three reflections. Furthermore, the orientation preserving isometries of $\mathbb{H}$ are products of exactly two reflections.
3. If L is an h-line and g is an hyperbolic isometry then $g(L)$ is an h-line.
4. Given any two h-lines L_1 and L_2, there is a ρ-isometry g such that $g(L_1) = L_2$.

A conformal (orientation preserving) isometry is of one of the three types: parabolic, elliptic or hyperbolic. The type of isometry can be recognized by the location of the fixed points or by the function trace.

1. **Parabolic isometries:** An isometry g is parabolic if and only it can be represented as $g = \sigma_1\sigma_2$ where σ_j is a reflection in the geodesic L_j and L_1 and L_2 are parallel geodesics. Using the trace this becomes $\mathrm{Trace}^2(g) = 4$.
2. **Elliptic isometries:** An isometry g is elliptic if and only if it can be represented as $g = \sigma_1\sigma_2$ where σ_j is the reflection in L_j and L_1 and L_2 intersect at a point w. $\mathrm{Trace}^2(g) \in [0, 4)$
3. **Hyperbolic isometries:** An isometry g is hyperbolic if and only if it can be represented as $g = \sigma_1\sigma_2$ where σ_j is a reflection in the geodesic L_j and L_1 and L_2 are disjoint and have L_0 as the common orthogonal geodesic. $\mathrm{Trace}^2(g) \in (4, +\infty)$

1.2 Riemann surfaces

A Riemann surface X is a topological space that locally is identical to the complex plane $\mathbb{C}$ (More formally it is a Hausdorff connected space). Each point on X has an open neighborhood homeomorphic to some open subset of $\mathbb{C}$.

Riemann surfaces are defined in order for the concept of analytic function and complex analytic function theory to extend without difficulty to them. A Riemann surface can be covered with some open subsets $\{U_j : j \in J\}$ and together with a homeomorphism φ_j from U_j onto an open subset of the complex plane. The pair of them (φ_j, U_j) is called a *chart* and the set of all charts $\{(\varphi_j, U_j) : j \in j\}$ is called an *atlas* of the Riemann surface. Each homeomorphism φ_j also satisfies that whenever two open subsets from the atlas intersects $U = U_i \cap U_j \neq \varnothing$ then

$$\varphi_j^{-1} \circ \varphi_i : \varphi_j(U) \to \varphi_i(U)$$

is an analytic (or meromorphic) map between the two plane sets $\varphi_j(U)$ and $\varphi_i(U)$ in the complex plane. The compatibility of the atlases is an equivalence relation and the equivalence class on atlases is called a complex structure on X.

Definition 1.2.1. Let X be a Riemann surface. A function $f : X \to \mathbb{C}$ is called *analytic* (or *meromorphic*), if for every chart (φ_i, U_i) on X, the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is analytic (or meromorphic) on $\varphi(U)$ in the usual sense.

It is possible to define analytic (or meromorphic) maps between Riemann surfaces. If X and Y are two Riemann surfaces with atlases $\{(\varphi_j, U_j) : j \in J\}$ and $\{(\psi_i, V_i) : i \in I\}$ then a continuous map $f : X \to Y$ is analytic (or meromorphic) if

$$\psi_i \circ f \circ \varphi_j^{-1} : \varphi_j(U_j \cap f^{-1}(V_i)) \to \mathbb{C}$$

is analytic. We say that the function $f : X \to Y$ is *holomorphic*.

Definition 1.2.2. Let X and Y be two Riemann surfaces. A function $f : X \to Y$ is called *biholomorphic* if it is bijective and both $f : X \to Y$ and $f^{-1} : Y \to X$ are holomorphic.

Two Riemann surfaces X and Y are isomorphic (or conformally equivalent) if there is a biholomorphic function f of X onto Y. We do not distinguish between such equivalent surfaces.

Example 1.2.1. There are a numerous examples of Riemann surfaces.

1. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The topology on $\widehat{\mathbb{C}}$ can be defined to be the open sets V in $\mathbb{C}$, together with the unions $V \cup \{\infty\}$ and with this topology $\widehat{\mathbb{C}}$ is a compact Haussdorff space homeomorphic to the 2-sphere. One possible atlas on $\widehat{\mathbb{C}}$ is to take $\{(\varphi_i, U_i)\}$ for $i = 1, 2$ to be $U_1 = \mathbb{C}$ and $U_2 = \widehat{\mathbb{C}} \backslash \{0\}$. The homeomorphisms in this case can be defined to be $\varphi_1 = 1_d$ and $\varphi_2 = 1/z$ for $z \in \mathbb{C}$ and $\varphi_2 = 0$ for $z = \infty$.

   Clearly, $\widehat{\mathbb{C}} = U_1 \cup U_2$ by definition, and φ_1, φ_2 are homeomorphisms. Their compositions $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$ and $\varphi_1 \circ \varphi_2^{-1}(z) = 1/z$ are analytic on $\varphi_1(U_1 \cap U_2) = \mathbb{C} \backslash \{0\}$ and $\varphi_2(U_1 \cap U_2) = \mathbb{C} \backslash \{0\}$ respectively.

Hence, the atlas is analytic and gives the desired complex structure on $\widehat{\mathbb{C}}$. The resulting Riemann surface is called the *Riemann sphere*.

2. If we take any surface (without singularities) and a (finite) group acting discontinuously on the surface. The quotient space will be a Riemann surface.

Riemann surfaces as quotient spaces

One method of constructing a Riemann surface is to form the quotient space with respect to a discontinuous group action. It is a well known fact that every Riemann surface arise in this way.

Theorem 1.2.1. *Every simply connected Riemann surface is conformally equivalent to just one of:*

1. *the Riemann sphere,* $\widehat{\mathbb{C}}$.
2. *the Euclidean plane,* $\mathbb{C}$.
3. *the hyperbolic plane,* $\mathbb{H}$.

Theorem 1.2.2. *[3] Let D be a subdomain of $\widehat{\mathbb{C}}$ and let G be a group of Möbius transformations which leaves D invariant and which acts discontinuously in D. Then D/G is a Riemann surface.*

There is a converse to the theorem 1.2.2 called the uniformization theorem, due to F. Klein and H. Poincaré (see [3], [17]):

Theorem 1.2.3. *(Uniformization theorem) A Riemann surface X is the quotient space of either the Riemann sphere, the Euclidean plane or the hyperbolic plane by a group Γ acting properly discontinuously on it.*

A Riemann surface X of genus $g \geq 2$ is the quotient space of the hyperbolic plane by a group Γ acting discontinuously on it; these groups are called Fuchsian groups. The projection $p : \mathbb{H} \to X$ is the universal covering of X, as quotient space. Now Γ is the group of deck transformations of the universal covering p and we say that Γ is the fundamental group of X.

1.3 Fuchsian groups

Definition 1.3.1. A group G of Möbius transformations is a Fuchsian group if and only if there is some G-invariant disc in which G acts discontinuously.

The notion of acting discontinuously means that for any compact subset U of the G-invariant disc we have $g(U) \cap U = \emptyset$, $g \in G$, except for a finite number of elements in G.

A Fuchsian group Γ has a presentation with generators

$$x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g, p_1, \ldots, p_s, h_1, \ldots, h_t$$

and relations

$$x_1^{m_1} = \ldots = x_r^{m_r} = \prod_{i=1}^{r} x_i \prod_{j=1}^{g} [a_j, b_j] \prod_{i=1}^{s} p_k \prod_{l=1}^{t} h_l = 1$$

The generators of the Fuchsian group are of 4 kinds

1. x_i are the elliptic elements.
2. a_j, b_j are the hyperbolic elements.
3. p_k are the parabolic elements.
4. h_l are the hyperbolic boundary elements.

The signature of such a Fuchsian group is defined to be

$$s(\Gamma) = (g; [m_1, \ldots, m_r], s, t) \tag{1.1}$$

where $m_i \geq 2$ are integers and are called the periods of Γ.
We are only interested in compact Riemann surfaces and from now on the Fuchsian groups will not contain parabolic or hyperbolic boundary elements. The elliptic elements in a Fuchsian group will be rotations with angle $2\pi/n$, with $n \in \mathbb{Q}$. Any elliptic element in Γ is conjugate to a power of x_i for $i = 1, \ldots, r$.
A Fuchsian group Γ containing only hyperbolic elements will be called a Fuchsian surface group. Its signature is $s(\Gamma) = (g; [-])$

Definition 1.3.2. A domain D of the hyperbolic plane is a *fundamental domain* for a Fuchsian group Γ if and only if

1. given $z \in \mathbb{H}$ there exists $g \in \Gamma$ such that $g(z) \in D$.
2. if there exist $z \in D$ and $1_d \neq g \in \Gamma$ such that $g(z)$ is also in D, then $z, g(z) \in \partial D$.
3. The hyperbolic area of ∂D is 0, $(\mu(\partial D) = 0)$.

If a Fuchsian group Γ is finitely generated, then one can choose a fundamental domain D for Γ which is homeomorphic to a disc and such that ∂D is a union of h-segments, called the sides. Moreover, there is a finite number of points in ∂D which divide ∂D in the above segments. In this case ∂D is called a *fundamental polygon* of Γ.
Two sides $\alpha, \alpha' \in \partial D$ are congruent if there exists $g \in \Gamma$ such that $\alpha = D \cap g(D)$, $\alpha' = D \cap g^{-1}(D)$ and $\alpha = g(\alpha')$. If g is an involution notice that α and α' lie on the same h-line.

If $D \cap g(D) \neq \emptyset$ for some $g \in \Gamma$ and $D \cap g(D)$ is not a common side, then $D \cap g(D)$ consists of a finite number of vertices.
From the above we get that conjugated vertices in D correspond to a conic point in $\mathbb{H}/\Gamma$.
Now, the generators of the Fuchsian group Γ pair the congruent sides of a fundamental polygon for Γ. The configuration of one such polygon (called a labeled polygon) is for the above presentation as follows:

$$\gamma_1\gamma_1' \ldots \gamma_r\gamma_r'\alpha_1\beta_1\alpha_1'\beta_1' \ldots \alpha_g\beta_g\alpha_g'\beta_g'$$

So the generator x_i pairs the sides γ_i and γ_i', a_i pairs the sides α_i and α_i' and b_i pairs the sides β_i and β_i'.
The vertex formed by γ_i and γ_i' has angle $2\pi/m_i$.
Let Γ be a Fuchsian group and let D be a fundamental domain for Γ. A Γ-tessellation of $\mathbb{H}$ is the configuration of $\mathbb{H}$ formed by D and its images under Γ.

The hyperbolic area for a Fuchsian group with signature (1.1) is the area of any of its fundamental domains, it equals:

$$\mu(\Gamma) = 2\pi\big(2g - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big)\big). \tag{1.2}$$

A fundamental domain D with the above identifications on its boundary has the structure of a compact Riemann surface.
Given a fundamental polygon ∂D satisfying the above conditions such that $\mu(D) > 0$, then there exists a Fuchsian group Γ with signature (1.1) having D has a fundamental domain.
Let Γ_1 be a subgroup of Γ of finite index N, then $\Gamma = \bigcup_1^N g_i(\Gamma_1)$, where $\{g_i\}$ is a transversal of Γ_1 in Γ. Then, if D is a fundamental domain for Γ, we get that $F_1 = \bigcup_1^N g_i(F)$ is a fundamental domain for Γ_1. That is, the monomorphism $i : \Gamma_1 \to \Gamma$ determines, via the transitive representation $\theta : \Gamma \to \Sigma_N$, the covering $f : \mathbb{H}/\Gamma_1 \to \mathbb{H}/\Gamma$. The map θ is called the monodromy of the covering f and $\theta(\Gamma)$ is called the monodromy group of the covering.
Now we get the Riemann-Hurwitz formula:

$$[\Gamma : \Gamma_1] = N = \frac{\mu(\Gamma_1)}{\mu(\Gamma)}. \tag{1.3}$$

By the above one can prove:

Theorem 1.3.1. *([21]) Let Γ have signature (1.1). Then Γ contains a subgroup Γ_1 of index N with signature*

$$(g'; [n_{11}, n_{12}, \ldots, n_{1\rho_1}, \ldots, n_{r1}, \ldots, n_{r\rho_r}]; s', t')$$

if and only if

(a) There exists a finite permutation group G, transitive on N points, and an epimorphism $\theta : \Gamma \to G$ satisfying the following relations:

(i) The permutation $\theta(x_j)$ has precisely ρ_j cycles of lengths less than m_j, the lengths of these cycles being $m_j/n_{j1}, \dots, m_j/n_{j\rho_j}$.

(ii) If we denote the number of cycles in the permutation $\theta(\gamma)$ by $\delta(\gamma)$ then

$$s' = \sum_{k=1}^{s} \delta(p_k) = \sum_{l=1}^{t} \delta(h_l).$$

(b) $\mu(\Gamma_1)/\mu(\Gamma) = N$.

Automorphism groups of Riemann surfaces

Definition 1.3.3. A conformal homeomorphism $f : X \to X$ is called an automorphism of X.

The set of automorphisms of X form a group under composition and we denote this group by $Aut(X)$. For the three simply connected Riemann surfaces given in the uniformization theorem (1.2.3) the automorphism groups are

Theorem 1.3.2. *[11]*

1. $Aut(\widehat{\mathbb{C}}) = PSL(2, \mathbb{C})$.
2. $Aut(\mathbb{C}) = \{z \mapsto az + b | a, b \in \mathbb{C}, a \neq 0\}$.
3. $Aut(\Delta) = PSL(2, \mathbb{R})$.

In each of these three cases the group consists of Möbius transformations. The group of automorphisms is not necessarily finite. However, for compact Riemann surfaces of genus $g \geq 2$ this is always the case.
Any Riemann surface is conformally equivalent to a Riemann surfaces uniformized by a Fuchsian surface group.
Given a Riemann surface X represented as the quotient space $\mathbb{H}/\Gamma$, with Γ a Fuchsian surface group, a finite group G is a group of automorphisms of X if and only if there exists a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$. The Fuchsian group Δ is the lifting of G to the universal covering $p : \mathbb{H} \to \mathbb{H}/\Gamma$ and is called the universal covering transformation group of (X, G). Notice that $\mathbb{H}/\Delta = X/G$. The covering $f : X \to \mathbb{H}/\Delta$ is a regular covering with monodromy group $G = \Delta/\Gamma$.
In our case we have the epimorphism $\theta : \Delta \to G$ and we are interested in finding the structure of $\theta^{-1}(G_1) = \Lambda$, with $G_1 \leq G$. But Λ is determined by the action of Δ on the Λ-cosets ($\rho : \Delta \to \Sigma_{|\Delta:\Lambda|}$), which is the same as the action of $G = \theta(\Delta)$ on the G_1-cosets. Topologically θ and ρ are the monodromy maps of the coverings $f : X \to X/G$ and $f_1 : X/G_1 \to X/G$. The structures of Λ and X/G_1 are determined by theorem 1.3.1.

Example 1.3.1. To see how the theory works, let Δ be a Fuchsian group with signature $s(\Delta) = (0; [3, 4, 6])$, then Δ has a presentation as follows:

$$\Delta = \langle x_1, x_2, x_3 | x_1^3 = x_2^4 = x_3^6 = x_1 x_2 x_3 = 1 \rangle.$$

Let $\langle 2, 3, 3 \rangle$ be the group of order 24 with presentation $\langle 2, 3, 3 \rangle = \langle s, t, a | s^4 = t^4 = (st)^4 = a^3 = 1, s^2 = t^2, a^2 sa = t, a^2 ta = st \rangle$.
Now consider the epimorphism

$$\theta : \Delta \to \langle 2, 3, 3 \rangle,$$

defined by $\theta(x_1) = sa$, $\theta(x_2) = s$ and $\theta(x_3) = a^2 t$,
Let $G_1 = \langle a \rangle$ be a subgroup of order 3 in $\langle 2, 3, 3 \rangle$ and $\Lambda = \theta^{-1}(\langle a \rangle)$.
The eight $\langle a \rangle$-cosets have representatives

$$1, s, t, st, s^2, s^3, t^3, ts,$$

and we label the cosets $1, 2, \ldots, 8$ in this order.
The action of $\theta(x_1) = sa$, on the $\langle a \rangle$-cosets has the cycle structure

$$(1, 2, 7)(3, 5, 6)(4)(8).$$

The action of $\theta(x_2) = s$, on the $\langle a \rangle$-cosets has the cycle structure

$$(1, 2, 5, 6)(3, 4, 7, 8).$$

The action of $\theta(x_3) = s^2 a$ on the $\langle a \rangle$-cosets has the cycle structure

$$(1, 4, 3, 5, 8, 7)(2, 6).$$

We get conical points (points whose angle sum does not add upp to 2π) with angles

$$3 \cdot 2\pi/3, 3 \cdot 2\pi/3, 2\pi/3, 2\pi/3$$

for the action of $\theta(x_1)$,

$$4 \cdot 2\pi/4, 4 \cdot 2\pi/4$$

for the action of $\theta(x_2)$ and

$$2 \cdot 2\pi/6 = 2\pi/3, 6 \cdot 2\pi/6$$

for the action of $\theta(x_3)$.
Thus, the subgroup Λ has three conical points of order 3 and the Riemann-Hurwitz formula gives $8(-2 + \frac{2}{3} + \frac{3}{4} + \frac{5}{6}) = (-2 + 2g + 3 \cdot \frac{2}{3})$
and so the genus g, of $\mathbb{H}/\Lambda$, is $g = 1$. Hence, the signature of Λ is $s(\Lambda) = (1; [3, 3, 3])$.

Theorem 1.3.3. *[11] Let X be a compact Riemann surface of genus $g \geq 2$. Then $|Aut(X)| \leq 84(g - 1)$.*

The finiteness was first proved by Schwartz in 1878 and the bound given above was proved by Hurwitz in 1893. The Riemann surfaces with automorphism groups of order $84(g-1)$ are called Hurwitz surfaces and their automorphism groups are called Hurwitz groups. Hurwitz surfaces are interesting since they are the surfaces admitting the maximal number of automorphisms.

Theorem 1.3.4. *([11]) A finite group H is a Hurwitz group if and only if H is non-trivial and has two generators x and y satisfying the relations*

$$x^2 = y^3 = (xy)^7 = 1.$$

It is a well-known fact that no Riemann surface of genus 2 admits an automorphism of order 7 ([2]). Therefore there are no Hurwitz surfaces of genus 2, neither Hurwitz groups of order 84. There is a unique Hurwitz surface of genus 3: the Klein quartic, its automorphism group is $PSL(2,7)$.

1.4 Trigonal Riemann surfaces

Let X_g be a compact Riemann surface of genus $g \geq 2$. The surface X_g can be represented as a quotient $X_g = \mathbb{H}/\Gamma$ of the hyperbolic plane under the action of a Fuchsian group Γ with no elliptic elements.

Definition 1.4.1. A Riemann surface X is called *trigonal* if it admits a three sheeted covering $f : X \to \widehat{\mathbb{C}}$ of the Riemann sphere. If f is a cyclic regular covering, then X is called a *cyclic trigonal* Riemann surface. A non-cyclic trigonal Riemann surface is said to be *generic trigonal*. The covering f is called *cyclic* respectively *generic trigonal morphism*.

If X is a cyclic trigonal Riemann surface then there exists an automorphism of order 3 $\varphi : X \to X$ such that $X/\langle\varphi\rangle$ is the sphere (with conical points). In general a Riemann surface X is called cyclic p-gonal (p prime) if X admits an automorphism $\varphi : X \to X$ of order p such that $X/\langle\varphi\rangle$ lies on the sphere.
The most well known p-gonal surfaces are for $p = 2$, the so called hyperelliptic surfaces. Cyclic p-gonal surfaces have equations of the type $y^p = c(x)$. The only cyclic 7-gonal Riemann surface of genus 3 is Klein's quartic with equation $y^7 = x(x^2+1)$. Klein's quartic is a surface of genus 3 with full group of automorphisms $Aut(KQ) = PSL(2,7)$, which is simple, so the 7-gonal morphisms are non-normal in $PSL(2,7)$.
Trigonal surfaces have been well studied by Accola ([1],[2]) and also by Costa and Izquierdo ([5], [6]). Accola showed in [1] that if the trigonal Riemann surface X has genus $g \geq 5$ the trigonal morphism is unique. Costa and Izquiredo showed in [5] that there are no trigonal Riemann surfaces of genus 3 with non-normal trigonal morphisms. The question is if there are trigonal Riemann surfaces of genus 4 with non-normal trigonal morphisms.

The connection between trigonal Riemann surfaces and Fuchsian groups is given by the following theorem

Theorem 1.4.1. *([6]) Let X be a Riemann surface. Then X admits a cyclic trigonal morphism if and only if there is a Fuchsian group Δ with signature $(0; [3, 3, \ldots, 3])$ and an index three normal surface subgroup Γ of Δ, such that Γ uniformizes X.*

There is a useful result from González ([9]) that we will use later.

Theorem 1.4.2. *If the automorphism group $Aut(X)$ of a Riemann surface X contains automorphisms τ_1, τ_2 of the same prime order and such that the quotient surfaces $X/\langle\tau_i\rangle$, $(i = 1, 2)$, are isomorphic to $\widehat{\mathbb{C}}$; then τ_1 and τ_2 are conjugate in $Aut(X)$.*

1.5 Teichmüller spaces

In this section we will define the Teichmüller space and the Moduli space. The use of these two space is to classify the geometry and the conformal structure of Riemann surfaces.
In fact, the Teichmüller space of a Fuchsian group classifies the geometries of a surface of genus g, and the Moduli space of a Fuchsian group classifies the conformal structures of a surface of genus g.

Definition 1.5.1. Let X be a Riemann surface of genus g and $Aut(X)$ its automorphism group and let G be a finite group. We say that G acts topologically on X if there exists a monomorphism $r : G \to Aut(X)$.

If $r' : G \to \widetilde{X}$ is a topological action on another Riemann surface, then we say that r and r' are topologically equivalent actions if there exists an $\omega \in Aut(G)$ and a homeomorphism $h : X \to \widetilde{X}$ such that

$$r'(g) = hr(\omega(g))h^{-1} \text{ for all } g \in G$$

Let Γ be a Fuchsian group with signature $(g; [m_1, \ldots, m_r])$ and $\Omega = Aut(\mathbb{H})$. We then define the Weil space as:

Definition 1.5.2. The *Weil space* of Γ with respect to Ω is the set

$$R(\Gamma) = \{ \text{ monomorphisms } r : \Gamma \to \Omega : r(\Gamma) \text{ is a Fuchsian group}\}.$$

The Teichmüller space of Γ is then defined to be:

Definition 1.5.3. The *Teichmüller space* of Γ is the orbit space

$$T(\Gamma) = R(\Gamma)/Aut(\Omega)$$

of $R(\Gamma)$ under the action of $Aut(\Omega)$, endowed with the quotient topology.

The main properties of the Teichmüller space are given by the following (see [18])

Theorem 1.5.1. *1. The topology of $T(\Gamma)$ can be derived from the Teichmüller metric, and $T(\Gamma)$ is a complete metric space of finite dimension $d(\Gamma)$.*

2. If Γ is a Fuchsian group with signature $s(\Gamma) = (g; [m_1, \ldots, m_r])$ then $T(\Gamma)$ is a cell of (complex) dimension $d(\Gamma) = 3g - 3 + r$. This was proved by Fricke and Klein [8]

3. Given two Fuchsian groups Γ and Γ' and a group monomorphism $\alpha : \Gamma \to \Gamma'$ the induced map

$$T(\alpha) : T(\Gamma) \to T(\Gamma') : [r] \mapsto [r\alpha],$$

is an isometric embedding [10].

We have seen that given $r \in R(\Gamma)$ and $\beta \in Aut(\Omega)$ then $r\beta \in R(\Gamma)$. Also, if $\alpha \in Aut(\Omega)$ and $s = \alpha r$, then $s\beta = a(r\beta)$ so there is an action

$$Aut(\Gamma) \times T(\Gamma) \to T(\Gamma) : (\beta, [r]) \mapsto [r\beta].$$

Definition 1.5.4. (Moduli space and modular group) The modular group $Mod(\Gamma)$ of Γ is the quotient $Mod(\Gamma) = Aut(\Gamma)/Inn(\Gamma)$, where $Inn(\Gamma)$ is the normal subgroup of $Aut(\Gamma)$ consisting of all inner automorphisms of Γ. The *moduli space* of Γ is the quotient $M(\Gamma) = T(\Gamma)/Mod(\Gamma)$ endowed with the quotient topology.

The main properties of the modular group are:

1. The map $Mod(\Gamma) \times T(\Gamma) \to T(\Gamma)$ given by $([\beta], [r]) \mapsto [r\beta]$ defines an action and $M(\Gamma) = T(\Gamma)/Mod(\Gamma)$.

2. $Mod(\Gamma)$ acts as a group of isometries on $T(\Gamma)$.

3. If $s(\Gamma) = (g; [-])$ then the elements of any subgroup of $Mod(\Gamma)$ have a common fixed point.

4. $Mod(\Gamma)$ acts as a discontinuous group of transformations of $T(\Gamma)$.

5. The action of $Mod(\Gamma)$ on $T(\Gamma)$ is not necessarily faithful, that is the the group homomorphism

$$Mod(\Gamma) \to Isom(T(\Gamma)) : [\beta] \to T(\beta)$$

is not necessarily injective.

Definition 1.5.5. A Fuchsian group Γ such that there does not exist any other Fuchsian group containing it with finite index is called a *finite maximal* Fuchsian group.

An important consequence was proved by Greenberg:

Theorem 1.5.2. *(See [10], [15]) The following conditions are equivalent:*

1. *$Mod(\Gamma)$ fails to act faithfully on $T(\Gamma)$.*
2. *There exists a Fuchsian group Γ' and a group monomorphism $\alpha : \Gamma \to \Gamma'$ such that $d(\Gamma) = d(\Gamma')$ and $\alpha(\Gamma)$ is a normal subgroup of Γ'.*

The full list such of pairs $(s(\Gamma), s(\Gamma'))$ of signatures of Fuchsian groups such was obtained by Singerman in [22].

$s(\Gamma)$	$s(\Gamma')$	$[\Gamma' : \alpha(\Gamma)]$
$(2; [-])$	$(0; [2,2,2,2,2,2])$	2
$(1; [t,t])$	$(0; [2,2,2,2,t])$	2
$(1; [t])$	$(0; [2,2,2,2t])$	2
$(0; [t,t,u,u])$	$(0; [2,2,t,u])$	2
$(0; [t,t,u])$	$(0; [2,t,2u])$	2
$(0; [t,t,t,t])$	$(0; [2,2,2,t])$	4
$(0; [t,t,t])$	$(0; [3,3,t])$	3
$(0; [t,t,t])$	$(0; [2,3,2t])$	6

Normal pairs of Fuchsian groups

To decide whether a given finite group can be the full group of automorphism of some compact Riemann surface we will need all pairs of signatures $s(\Gamma)$ and $s(\Gamma')$ for some Fuchsian groups Γ and Γ' such that $\Gamma \leq \Gamma'$ and $d(\Gamma) = d(\Gamma')$. The full list in the non-normal case was also obtained by Singerman in [22].

$s(\Gamma)$	$s(\Gamma')$	$[\Gamma' : \alpha(\Gamma)]$
$(0; [7,7,7])$	$(0; [2,3,7])$	24
$(0; [2,7,7])$	$(0; [2,3,7])$	9
$(0; [3,3,7])$	$(0; [2,3,7])$	8
$(0; [4,8,8])$	$(0; [2,3,8])$	12
$(0; [3,8,8])$	$(0; [2,3,8])$	10
$(0; [9,9,9])$	$(0; [2,3,9])$	12
$(0; [4,4,5])$	$(0; [2,4,5])$	6
$(0; [n,4n,4n])$	$(0; [2,3,4n])$	6
$(0; [n,2n,2n])$	$(0; [2,4,2n])$	4
$(0; [n,3,3n])$	$(0; [2,3,3n])$	4
$(0; [n,2,2n])$	$(0; [2,3,2n])$	3

Non-normal pairs of Fuchsian signatures

Chapter 2

Trigonal Riemann surfaces of genus 4

First of all we begin with finding the possible signatures of the Fuchsian groups unifomizing the quotient surface X_4/G, where X_4 is a cyclic trigonal Riemann surface and $G = Aut(X_4)$. These signatures will be given in lemma 2.1.1.
Only the maximal signatures will give the full automorphism groups of the Riemann surfaces. These will be given in lemma 2.1.2.
After having obtained the signatures, the next thing is to identify the possible automorphism groups of each order (the order being a multiple of 3). This is done by finding an epimorphism from the Fuchsian group having one of the maximal signatures onto a group of the respective order.
Once we can find an epimorphism, we have to see how the image of each generator under the epimorphism acts on the cosets of subgroups of order 3 in the automorphism group.
The action of the images of the generators will give us the signature of $\Lambda = \theta^{-1}(\langle\tau\rangle)$. It Λ uniformizes the Riemann sphere with exactly 6 conical points we know by theorem 1.4.1 that the surface obtained is a trigonal surface.
The method is skissed in the following diagram

$$\begin{array}{rcl} \theta : \Delta & \longrightarrow & G \\ \uparrow & & \uparrow \\ \Lambda & \longrightarrow & \langle\tau\rangle \\ 3{:}1 \uparrow & & \uparrow \\ \Gamma & \longrightarrow & 1_d, \end{array}$$

where the trigonal Riemann surface is uniformized by $\Gamma = ker(\theta)$, and the signature of $\Lambda = \theta^{-1}(\langle\tau\rangle)$ is $(0; [3, 3, 3, 3, 3, 3])$.

2.1 Developing the signatures

Lemma 2.1.1. *Let X_4 be a cyclic trigonal Riemann surface, admitting a trigonal morphism onto the sphere, $f : X_4 \to \widehat{\mathbb{C}}$ with 6 conical points. Then the Fuchsian group Δ uniformizing the surface X/G can have one of the following signatures:*

Signatures			
Order	Signature	Order	Signature
3	$(0; [3,3,3,3,3,3])$	27	$(0; [3,3,9])^*$
6	$(0; [3,3,6,6])^*$	30	$(0; [2,5,10])^*$
	$(0; [2,6,6,6])$	36	$(0; [3,4,4])^*$
	$(0; [2,2,3,3,3])$		$(0; [3,3,6])^*$
	$(0; [2,2,2,3,6])$		$(0, [2,6,6])^*$
	$(0; [2,2,2,2,2,2])$		$(0; [2,4,12])$
9	$(0; [9,9,9])^*$		$(0; [2,2,2,3])$
	$(0; [3,3,3,3])^*$	42	$(0; [2,3,42])$
12	$(0; [6,6,6])^*$	45	$(0; [3,3,5])^*$
	$(0; [4,6,12])$	48	$(0; [2,4,8])^*$
	$(0; [3,12,12])^*$		$(0; [2,3,24])$
	$(0; [2,3,3,3])$	54	$(0; [2,3,18])$
	$(0; [2,2,4,4])^*$	60	$(0; [2,5,5])^*$
	$(0; [2,2,3,6])$		$(0; [2,3,15])$
	$(0; [2,2,2,2,2])$	72	$(0; [3,3,4])^*$
15	$(0; [5,5,5])^*$		$(0; [2,4,6])$
	$(0; [3,5,15])^*$		$(0; [2,3,12])$
18	$(0; [3,6,6])^*$	90	$(0; [2,3,10])$
	$(0; [2,9,18])^*$	108	$(0; [2,3,9])$
	$(0; [2,2,3,3])^*$	120	$(0; [2,4,5])$
	$(0; [2,2,2,6])$	144	$(0; [2,3,8])$
21	$(0; [3,3,21])^*$	252	$(0; [2,3,7])$
24	$(0; [4,4,4])^*$		
	$(0; [3,4,6])$		
	$(0; [3,3,12])^*$		
	$(0; [2,8,8])^*$		
	$(0; [2,6,12])^*$		
	$(0; [2,2,2,4])$		

*Non-maximal group

Proof. Since X_4 is a trigonal Riemann surface uniformized by a Fuchsian group Γ, it will admit the cyclic trigonal morphism if and only if

- there is a maximal Fuchsian group Δ with signature $(0; [m_1, \ldots, m_r])$.
- there is a finite group $G = Aut(X_4)$ such that $\langle\varphi\rangle \leq G$.
- there is an epimorphism $\theta : \Delta \to G$ with $Ker(\theta)$ such that $\Lambda = \theta^{-1}(\langle\varphi\rangle)$ is a Fuchsian group with signature $(0; [3,3,3,3,3,3])$.

$$
\begin{array}{ccl}
X_4 \cong \mathbb{H}/\Gamma & & \\
\downarrow & \searrow & \\
& & \mathbb{H}/\Lambda = X_4/\langle\varphi\rangle \\
& & s(\Lambda) = (0; [3,3,3,3,3,3]) \\
\downarrow & \swarrow & \\
X_4/G \cong \mathbb{H}/\Delta & &
\end{array}
$$

Let $\theta : \Delta \to G$ be such an epimorphism, then $s(\Delta) = (0; [m_1, \ldots, m_r])$ where the m_i runs over the divisors of $|G|$. Now applying the Riemann-Hurwitz formula (1.2) we get

$$\frac{\mu(\Lambda)}{\mu(\Delta)} = [\Delta : \Lambda] = \frac{|G|}{|\langle\varphi\rangle|} = \frac{|G|}{3} \tag{2.1}$$

and so we get

$$3\mu(\Lambda) = |G|\mu(\Delta). \tag{2.2}$$

Now using the formula for calculating the hyperbolic area for the fundamental region of a Fuchsian group we get

$$3(-2 + \sum_{i=1}^{6}(1 - \frac{1}{3})) = |G|(-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})). \tag{2.3}$$

That is

$$6 = |G|(-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})). \tag{2.4}$$

To find the signatures it is sufficient to solve the integer equations that arises from equation (2.4). Let $x_1, \ldots, x_r$ be the multiple of each divisor then we can write (2.4) as

$$6 = |G|(-2 + \sum_{i=1}^{r} x_i(1 - \frac{1}{m_i})). \tag{2.5}$$

The different solutions to (2.5) are given in the statement of the theorem. □

By comparison with Singerman's list of maximal Fuchsian groups [22] we get the following list of maximal Fuchsian groups:

Lemma 2.1.2. *Let X_4 be a cyclic trigonal Riemann surface, admitting a trigonal morphism onto the sphere, $f : X_4 \to \widehat{\mathbb{C}}$ with 6 conical points. Then the* ***maximal*** *Fuchsian group Δ uniformizing the surface X/G can have one of the following signatures:*

Signatures			
Order	Signature	Order	Signature
3	$(0; [3, 3, 3, 3, 3, 3])$	36	$(0; [2, 4, 12])$
6	$(0; [2, 6, 6, 6])$		$(0; [2, 2, 2, 3])$
	$(0; [2, 2, 3, 3, 3])$	42	$(0; [2, 3, 42])$
	$(0; [2, 2, 2, 3, 6])$	48	$(0; [2, 3, 24])$
	$(0; [2, 2, 2, 2, 2, 2])$	54	$(0; [2, 3, 18])$
12	$(0; [4, 6, 12])$	60	$(0; [2, 3, 15])$
	$(0; [2, 3, 3, 3])$	72	$(0; [2, 4, 6])$
	$(0; [2, 2, 3, 6])$		$(0; [2, 3, 12])$
	$(0; [2, 2, 2, 2, 2])$	90	$(0; [2, 3, 10])$
18	$(0; [2, 2, 2, 6])$	108	$(0; [2, 3, 9])$
24	$(0; [3, 4, 6])$	120	$(0; [2, 4, 5])$
	$(0; [2, 2, 2, 4])$	144	$(0; [2, 3, 8])$
		252	$(0; [2, 3, 7])$

These maximal signatures will give us the full automorphism groups of the Riemann surfaces. Hence, while calculating the epimorphisms the important ones will be the ones from maximal Fuchsian groups, i.e. the ones from the above list.

2.2 Automorphism groups

We will now calculate suitable epimorphisms $\Delta \to G$ for groups G of order a multiple of 3.
Let τ be an element of order 3 in G. The surface $\mathbb{H}/ker(\theta)$ is trigonal if and only if $s(\Lambda) = s(\theta^{-1}(\langle\tau\rangle) = (0; [3, 3, 3, 3, 3, 3])$.

Proposition 2.2.1. *There are cyclic trigonal Riemann surfaces of genus 4. They form a subspace $\mathcal{M}_4^3$ of dimension 3 of the moduli space $\mathcal{M}_4$ of surfaces of genus 4.*

Proof. By lemma 2.1.1 there is only one signature

$$s(\Delta) = (0; [3, 3, 3, 3, 3, 3])$$

and there is a possible epimorphism to C_3 by sending $x_{2i+1} \mapsto t$ and $x_{2i} \mapsto t^{-1}$. In this case each x_i acts on the only coset (C_3) so there are 6 conical points of order 3 and the surface given by the signature $s(\theta^{-1}\langle\tau\rangle) = (0; [3, 3, 3, 3, 3, 3])$ is trigonal and the trigonal morphism is unique.
Using the dimension formula from theorem 1.5.1 for Teichmüller spaces we get that the dimension is $d(\Delta) = g - 3 + r = 0 - 3 + 6 = 3$.
By González's result [9] the space $\mathcal{M}_4^3$ has two disconnected components. One with the stabilizers of all the conical points rotating in the same direction and the other with half the stabilizers rotating in each direction. □

Proposition 2.2.2. *The subspace of $\mathcal{M}_4^3$ formed by trigonal Riemann surfaces with automorphism groups of order 6, has dimension 2. The automorphism group of the trigonal Riemann surface is either C_6 or D_3.*

Proof. Lemma 2.1.1 gives the following 5 signatures

$$\begin{array}{l} s(\Delta_1) = (0;[3,3,6,6]), \Delta_1 \overset{2}{\hookrightarrow} \Delta(0;[2,2,3,6]) \\ s(\Delta_2) = (0;[2,6,6,6]) \\ s(\Delta_3) = (0;[2,2,3,3,3]) \\ s(\Delta_4) = (0;[2,2,2,3,6]) \\ s(\Delta_5) = (0;[2,2,2,2,2,2]) \end{array}$$

and $G \cong C_6 = \langle u | u^6 = 1\rangle$ or $G \cong D_3 = \langle s,t | s^2 = t^3 = (st)^2\rangle$. C_6 and D_3 contain a unique subgroup of order 3 (hence they are normal), and they are generated by the elements $u^{\pm 2}$ in C_6 and $t^{\pm 1}$ in D_3.

Δ_2: Again, there is no epimorphism from Δ_2 to D_3 since there are no elements of order 6 in D_3.

For $\Delta(0;[2,6,6,6]) \to C_6$ there is a possible epimorphism since the elements of order 2 are u^3. u^3 having order 2 can not give any conical points, but again the elements of order 6 give 1 conical point of order 3 each. The Riemann-Hurwitz formula gives $2(-2+\frac{1}{2}+3\cdot\frac{5}{6}) = (-2+2g+3\cdot\frac{2}{3})$ and so $g=1$. Thus, the signature is $s(\theta^{-1}\langle\tau\rangle) = (1;[3,3,3])$ and the surface $\mathbb{H}/ker(\theta)$ is not trigonal.

Δ_3: For $\Delta(0;[2,2,3,3,3]) \to D_3$ it is possible to choose several ways of doing the epimorphism, but the action of each one of them is equivalent. The elements of order 2 give no conical points, but the elements of order 3 give 2 conical points each so there are 6 conical points. Thus, the signature is $s(\theta^{-1}\langle\tau\rangle) = (0;[3,3,3,3,3,3])$ and the surface $\mathbb{H}/ker(\theta)$ is trigonal and the trigonal morphism is unique.

For $\Delta(0;[2,2,3,3,3]) \to C_6$ there is a possible epimorphism and the 3 elements of order 3 give 6 conical points of order 3. Thus, the signature is $s(\theta^{-1}\langle\tau\rangle) = (0;[3,3,3,3,3,3])$, the surfaces $\mathbb{H}/ker(\theta)$ are trigonal and the trigonal morphisms are unique.

Δ_4: There is no epimorphism from Δ_4 to D_3 since there are no elements of order 6 in D_3.

For $\Delta(0;[2,2,2,3,6]) \to C_6$ there is a possible epimorphism and the only elements giving conical points are the elements of order 3 and 6. Hence there are 3 conical points of order 3 and the Riemann-Hurwitz formula gives $2(-2+\frac{1}{2}+3\cdot\frac{5}{6}) = (-2+2g+3\cdot\frac{2}{3})$ and so $g=1$. Thus, the signature is $s(\theta^{-1}\langle\tau\rangle) = (1;[3,3,3])$ and the surface $\mathbb{H}/ker(\theta)$ is not trigonal.

Δ_5: There is no epimorphism from Δ_5 to C_6 since the elements of order 2 can not generate C_6.

For $\theta : \Delta_5 \to D_3$ it is possible to find an epimorphism but the action of each x_i on the cosets will only permute the two of them so there will be no conical points. Hence, the signature is $s(\theta^{-1}(\langle\tau\rangle) = (2; [-])$ and the surface $\mathbb{H}/ker(\theta)$ is not trigonal.

Summing up, the only Fuchsian group that will give trigonal Riemann surfaces with automorphism groups of order 6 is the group with signatures $(0; [2, 2, 3, 3])$. The automorphism group is either C_6 or D_3.
Moreover, by the dimension formula (thm 1.5.1) the dimension of the subspace in $\mathcal{M}_4^3$ that these Riemann surfaces form is 2.

□

Proposition 2.2.3. *The subspace of $\mathcal{M}_4^3$ formed by cyclic trigonal Riemann surfaces with automorphism groups of order 12 has dimension 1. The automorphism group of the trigonal Riemann surface is either $C_6 \times C_2$ or D_6.*

Proof. Lemma 2.1.1 gives the following 6 signatures:

$$\begin{aligned}
&s(\Delta_1) = (0; [6, 6, 6]), \Delta_1 \overset{2}{\hookrightarrow} \Delta(0; [2, 6, 12]) \overset{3}{\hookrightarrow} \Delta(0; [2, 3, 12])\\
&s(\Delta_2) = (0; [4, 6, 12])\\
&s(\Delta_3) = (0; [3, 12, 12]), \Delta_3 \overset{2}{\hookrightarrow} \Delta(0; [2, 6, 12]) \overset{3}{\hookrightarrow} \Delta(0; [2, 3, 12])\\
&s(\Delta_4) = (0; [2, 3, 3, 3])\\
&s(\Delta_5) = (0; [2, 2, 4, 4]), \Delta_5 \overset{2}{\hookrightarrow} \Delta(0; [2, 2, 2, 4])\\
&s(\Delta_6) = (0; [2, 2, 3, 6])\\
&s(\Delta_7) = (0; [2, 2, 2, 2, 2])
\end{aligned}$$

and there are 5 groups of order 12 namely

1. $C_{12} = \langle u | u^{12} \rangle$
2. $C_6 \times C_2 = \langle s, t | s^6 = t^3 = [s, t] = 1 \rangle$
3. $D_6 = \langle s, t | s^6 = t^2 = (st)^2 = 1 \rangle$
4. $A_4 = \langle s, t | s^3 = t^3 = (st)^2 = 1 \rangle$
5. $T = \langle s, t | t^3 = s^4 = 1, s^3 ts = t^2 \rangle$

Δ_2: 1) $\Delta(0; [4, 6, 12]) \to C_{12}$; The elements of order 4 in C_{12} are $u^{\pm 3}$, the elements of order 6 are $u^{\pm 2}$ and the elements of order 12 are $u^{\pm 1}$ and $u^{\pm 5}$. There is a possible epimorphism but the elements of order 4 give no conical points, the elements of order 6 give 2 conical points and the elements of order 12 can only give one conical point of order 3 when acting in the $\langle u^4 \rangle$-cosets. The Riemann-Hurwitz formula gives $4(-2+\frac{3}{4}+\frac{5}{6}+\frac{11}{12}) = -2+2g+3\cdot\frac{2}{3}$ so $g = 1$. Thus, the signature is $s(\theta^{-1}\langle\tau\rangle) = (1; [3, 3, 3])$ and the surface $\mathbb{H}/ker(\theta)$ is not trigonal.

2,3,4,5) There are no elements of order 12 in any of $C_6 \times C_2$, D_6, A_4 or T. Hence there can not be any epimorphism.

Δ_4: 1) $\Delta(0;[2,3,3,3]) \to C_{12}$; The elements of order 3 are $u^{\pm 4}$. The product of 3 elements of order 3 is either u^4 or the identity, hence there can not be any epimorphism $\theta : \Delta_4 \to C_{12}$.

2) $\Delta(0;[2,3,3,3]) \to C_6 \times C_2$; The elements of order 2 in $C_6 \times C_2$ are t, s^3 or ts^3 and the elements of order 3 are $s^{\pm 2}$. The product of 3 elements of order 3 is either $s^{\pm 2}$ or the identity. The total product of them can never be the identity. Hence there can not be any epimorphism $\theta : \Delta_4 \to C_6 \times C_2$.

3) $\Delta(0;[2,3,3,3]) \to D_6$; The elements of order 3 are $s^{\pm 2}$. The product of 3 elements of order 3 is again either $s^{\pm 2}$ or the identity so there can not be any epimorphism $\theta : \Delta_4 \to D_6$.

4) $\Delta(0;[2,3,3,3]) \to A_4$; There exists an epimorphism in this case. There is one conjugacy class of elements of order 3 with representant s . The action of each element of order 2 on the $\langle s \rangle$-cosets gives no conical point. The action of each element of order 3 on the $\langle s \rangle$-cosets gives only one conical point of order 3 so the epimorphism $\theta : \Delta_4 \to A_4$ does not give a trigonal morphism and the surface $\mathbb{H}/ker(\theta)$ is not trigonal.

5) $\Delta(0;[2,3,3,3]) \to T$; The element of order 2 in $C_3 \rtimes C_4$ is s^2 and the elements of order 3 are $t^{\pm 1}$ so the total product is either s^2 or $s^2 t^{\pm 1}$ and so there can not be any epimorphism $\theta : \Delta_4 \to T$.

Δ_6: 1) $\Delta(0;[2,2,3,6]) \to C_{12}$; The product of the first 2 elements is the identity and the product of an element of order 3 and an element of order 6 can never be the identity. Hence there can not be any epimorphism $\theta : \Delta_6 \to C_{12}$.

2) $\Delta(0;[2,2,3,6]) \to C_6 \times C_2$; There are epimorphisms in this case, for example:

$$\left.\begin{array}{l} x_1 \mapsto s^3 t \\ x_2 \mapsto t \\ x_3 \mapsto s^2 \\ x_4 \mapsto s \end{array}\right\} \theta(x_1 x_2 x_3 x_4) = 1_d$$

The elements of order 2 do not give any conical points. Any element of order 3 gives 2 conical points of order 3 and an element of order 6 gives 4 conical point. In total there are 6 conical points of order 3 and the signature of Λ is $(0;[3,3,3,3,3,3])$. The surface $\mathbb{H}/ker(\theta)$ is trigonal and the trigonal morphism is normal.

3) $\Delta(0;[2,2,3,6]) \to D_6$; It is easy to check that there are epimorphisms in this case (for example $\theta(x_1) = st$, $\theta(x_2) = t$, $\theta(x_3) = s^4$ and $\theta(x_4) = s$). The subgroup generated by s^2 is normal in D_6. The elements of order 2 will give no conical points, an element of order 3 gives 4 conical points and

an element of order 6 gives 2 conical points of order 3. Hence, $s(\theta^{-1}(\langle s^2\rangle) = (0;[3,3,3,3,3,3])$ and the surface $\mathbb{H}/ker(\theta)$ is trigonal with normal trigonal morphism.

4) There are no elements of order 6 in A_4 so there can not be any epimorphism $\theta : \Delta_6 \to A_4$.

5) $\Delta(0;[2,2,3,6]) \to T$; The element of order 2 is s^2, the elements of order 3 are $t^{\pm1}$ and the elements of order 6 are ts^2, t^2s^2. $\theta(x_1x_2x_3) = t^{\pm1}$ so there can not be any epimorphism $\theta : \Delta_6 \to T$.

Δ_7: Because all the orders of the generators in $(0;[2,2,2,2,2])$ are coprime to 3, the signature $(0;[2,2,2,2,2])$ can not give any trigonal morphism.

Summing up we have: The only Fuchsian group that will give trigonal Riemann surfaces with automorphism groups of order 12 is the group with signature $(0;[2,2,3,6])$. The automorphism group is either $C_6 \times C_2$ or D_6. By the dimension formula (thm 1.5.1) the dimension of the subspace of $\mathcal{M}_4^3$ that they form has dimension 1.

□

Proposition 2.2.4. *There is no Riemann surface of genus 4 with automorphism group of order 18.*

Proof. For $|G| = 18$, lemma 2.1.1 gives the following 4 signatures:

$$s(\Delta_1) = (0;[3,6,6]), \Delta_1 \overset{4}{\hookrightarrow} \Delta(0;[2,4,6])$$
$$s(\Delta_2) = (0;[2,9,18]), \Delta_2 \overset{3}{\hookrightarrow} \Delta(0;[2,3,18])$$
$$s(\Delta_3) = (0;[2,2,3,3]), \Delta_3 \overset{2}{\hookrightarrow} \Delta(0;[2,2,2,3])$$
$$s(\Delta_4) = (0;[2,2,2,6])$$

There are 5 groups of order 18 namely

1. $C_{18} = \langle u | u^{18} = 1\rangle$
2. $C_3 \times C_6 = \langle s,t | s^3 = t^6 = [s,t] = 1\rangle$
3. $C_3 \times D_3 = \langle s,t | s^3 = t^6 = 1, t^5st = s^2\rangle$
4. $D_9 = \langle s,t | s^2 = t^9 = (st)^2 = 1\rangle$
5. $\langle 3,3,3,2\rangle = \langle a,b,t | a^3 = b^3 = t^2 = [a,b] = 1, tat = a^{-1}, tbt = b^{-1}\rangle$

Δ_1: 1) $\Delta(0;[3,6,6]) \to C_{18}$; The elements of order 3 and 6 in C_{18} all are powers of u^3, hence there can not be any epimorphism.

2) $\Delta(0;[3,6,6]) \to C_3 \times C_6$; There are 4 classes of subgroups of order 3, namely, $\langle s\rangle$, $\langle t^2\rangle$, $\langle st^2\rangle$ and $\langle st^4\rangle$.

The choices for the epimorphisms are:

$$\theta \begin{cases} x_1 \mapsto s^{\pm1}, t^{\pm2}, (st^2)^{\pm1}, (st^4)^{\pm1} \\ x_2 \mapsto t^{\pm1}, s^{\pm1}t^{\pm3}, s^{\pm1}t^{\pm1} \\ x_3 \mapsto t^{\pm1}, s^{\pm1}t^{\pm3}, s^{\pm1}t^{\pm1} \end{cases}.$$

Thus, there are several choices one can make. For example

$$\theta \begin{cases} x_1 \mapsto s \\ x_2 \mapsto st \\ x_3 \mapsto st^5 \end{cases}.$$

However, the elements of order 6 $t^{\pm1}, s^{\pm1}t^{\pm3}, s^{\pm1}t^{\pm1}$ will all become elements of order 3 when they are squared. Hence the action of an element of order 6 will give three conic points of order 3 exactly when acting on its square's cosets. An element of order 3 leaves six conic points of order 3 exactly when acting on its own cosets.

So each epimorphism will give one normal trigonal morphism, two automorphisms of order 3 with quotient surface of genus 1 and one automorphism of order 3 with quotient surface of genus 2.

3) $\Delta(0;[3,6,6]) \to C_3 \times D_3$; In this case there are 2 types of epimorphisms:

$$\theta_1 \begin{cases} x_1 \mapsto s \\ x_2 \mapsto t \\ x_3 \mapsto st^{-1} \end{cases}$$

and

$$\theta_2 \begin{cases} x_1 \mapsto st^2 \\ x_2 \mapsto s^2t^5 \\ x_3 \mapsto t^5 \end{cases}$$

Now, there are three conjugacy classes of subgroups of order 3 with representatives s, t^2 and st^2 and their cosets are given by

	$\langle \tau_1 = s \rangle$	$\langle \tau_2 = t^2 \rangle$	$\langle \tau_3 = st^2 \rangle$
1	$\{1, s, s^2\}$	$\{1, t^2, t^4\}$	$\{1, st^2, s^2t^4\}$
2	$\{t, st, s^2t\}$	$\{t, t^3, t^5\}$	$\{t, st^3, s^2t^5\}$
3	$\{t^2, st^2, s^2t^2\}$	$\{s, st^2, st^4\}$	$\{s, s^2t^2, t^4\}$
4	$\{t^3, st^3, s^2t^3\}$	$\{st, st^3, st^5\}$	$\{st, s^2t^3, t^5\}$
5	$\{t^4, st^4, s^2t^4\}$	$\{s^2, s^2t^2, s^2t^4\}$	$\{s^2, t^2, st^4\}$
6	$\{t^5, st^5, s^2t^5\}$	$\{s^2t, s^2t^3, s^2t^5\}$	$\{s^2t, t^3, st^5\}$

The action of each generator becomes:

		$\langle \tau = s \rangle$	$\langle \tau = t^2 \rangle$	$\langle \tau = st^2 \rangle$
θ_1				
s	$\rightsquigarrow$	$(1)(2)(3)(4)(5)(6)$	$(1,3,5)(2,6,4)$	$(1,3,5)(2,6,4)$
t	$\rightsquigarrow$	$(1,2,3,4,5,6)$	$(1,2)(3,4)(5,6)$	$(1,2,5,6,3,4)$
st^5	$\rightsquigarrow$	$(1,6,5,4,3,2)$	$(1,4)(2,5)(3,6)$	$(1,6,3,2,5,4)$
θ_2				
st^2	$\rightsquigarrow$	$(1,3,5)(2,6,4)$	$(1,3,5)(2,6,4)$	$(1)(2,4,6)(3)(5)$
s^2t^5	$\rightsquigarrow$	$(1,6,5,4,3,2)$	$(1,6)(2,3)(4,5)$	$(1,2,3,4,5,6)$
t^5	$\rightsquigarrow$	$(1,6,5,4,3,2)$	$(1,2)(3,4)(5,6)$	$(1,4,3,6,5,2)$

Hence, θ_1 gives two non-conjugated and normal trigonal morphisms, and one class of automorphsims of order 3 with quotient surface of genus 2.

θ_2 gives one normal trigonal morphism, one class of automorphisms of order 3 with quotient surface of genus 1 and one normal automorphism of order 3 with quotient surface of genus 2.

4,5) There are no elements of order 6 in D_9 and $\langle 3,3,3,2\rangle$.

Δ_3: 1) $\Delta(0;[2,2,3,3]) \to C_{18}$; The elements of order 2 and 3 in C_{18} all are powers of u^3, therefore, there can not be any epimorphism.

2) $\Delta(0;[2,2,3,3]) \to C_3 \times C_6$;

$$\left.\begin{array}{l} x_1 \mapsto t^3 \\ x_2 \mapsto t^3 \\ x_3 \mapsto s^{\pm1}, t^{\pm2}, s^{\pm1}t^{\pm2} \\ x_4 \mapsto s^{\pm1}, t^{\pm2}, s^{\pm1}t^{\pm2} \end{array}\right\} \theta(x_1x_2x_3x_4) = 1_d$$

For the product to become the identity, x_3 and x_4 need to be mapped to elements in $C_3 \times C_6$ whose product is the identity. Hence $\theta(\Delta_3) \subsetneq C_3 \times C_6$ (in fact $\theta(\Delta_3) = C_6$), and so there can be no such epimorphism.

3) $\Delta(0;[2,2,3,3]) \to C_3 \times D_3$;

$$\left.\begin{array}{l} x_1 \mapsto s^it^3 \\ x_2 \mapsto s^jt^3 \\ x_3 \mapsto s^{\pm1}, t^{\pm2}, s^{\pm1}t^{\pm2} \\ x_4 \mapsto s^{\pm1}, t^{\pm2}, s^{\pm1}t^{\pm2} \end{array}\right\} \theta(x_1x_2x_3x_4) = 1_d$$

Since $s^it^3s^jt^3 = s^{i+2j}$, there are 2 possible types of epimorphisms:

$$\theta_1 \left\{\begin{array}{l} x_1 \mapsto st^3 \\ x_2 \mapsto s^2t^3 \\ x_3 \mapsto st^2 \\ x_4 \mapsto t^4 \end{array}\right. \quad \theta_2 \left\{\begin{array}{l} x_1 \mapsto t^3 \\ x_2 \mapsto t^3 \\ x_3 \mapsto s^2t^2 \\ x_4 \mapsto st^4 \end{array}\right.$$

The action of these elements on the cosets is given by

		$\langle\tau = s\rangle$	$\langle\tau = t^2\rangle$	$\langle\tau = st^2\rangle$
θ_1				
st^3	$\rightsquigarrow$	$(1,4)(2,5)(3,6)$	$(1,4)(2,5)(3,6)$	$(1,2)(3,4)(5,6)$
s^2t^3	$\rightsquigarrow$	$(1,4)(2,5)(3,6)$	$(1,6)(2,3)(4,5)$	$(1,4)(2,5)(3,6)$
st^2	$\rightsquigarrow$	$(1,3,5)(2,4,6)$	$(1,3,5)(2,6,4)$	$(1)(2,4,6)(3)(5)$
t^4	$\rightsquigarrow$	$(1,5,3)(2,6,4)$	$(1)(2)(3)(4)(5)(6)$	$(1,3,5)(2,4,6)$
θ_2				
t^3	$\rightsquigarrow$	$(1,4)(2,5)(3,6)$	$(1,2)(3,4)(5,6)$	$(1,6)(2,3)(4,5)$
t^3	$\rightsquigarrow$	$(1,4)(2,5)(3,6)$	$(1,2)(3,4)(5,6)$	$(1,2)(3,4)(5,6)$
s^2t^2	$\rightsquigarrow$	$(1,3,5)(2,4,6)$	$(1,5,3)(2,4,6)$	$(1,3,5)(2)(4)(6)$
s^2t^4	$\rightsquigarrow$	$(1,5,3)(2,6,4)$	$(1,3,5)(2,4,6)$	$(1,5,3)(2)(4)(6)$

Hence, θ_1 gives one normal trigonal morphisms, one class of automorphsims of order 3 with quotient surface of genus 1 and one normal automorphism of order 3 with quotient surface of genus 2.

θ_2 gives one class of non-normal trigonal morphisms and two normal automorphisms of order 3 with quotient surface of genus 2.

4) $\Delta(0;[2,2,3,3]) \to D_9$;

$$\left.\begin{array}{l} x_1 \mapsto st^i \\ x_2 \mapsto st^j \\ x_3 \mapsto t^{\pm 3} \\ x_4 \mapsto t^{\pm 3} \end{array}\right\}$$

$\theta(x_1x_2x_3) = st^ist^jt^{\pm3}t^{\pm3} = t^{j-i}t^{\pm3}t^{\pm3} = t^{j-i}$ or $t^{j-i\pm3}$
Hence $\theta(x_1x_2x_3) = 1_d$ if $j-i \equiv 0 (\text{mod } 6)$ or $j-i \equiv \pm3 (\text{mod } 6)$.
Thus, $\theta(\Delta_3) \subsetneqq D_9$ (in fact $\theta(\Delta_4) = D_3$), and so there can not be any epimorphism.

5) $\Delta(0;[2,2,3,3]) \to \langle 3,3,3,2\rangle$;

$$\left.\begin{array}{l} x_1 \mapsto a^ib^jt \\ x_2 \mapsto a^kb^lt \\ x_3 \mapsto a^mb^n \\ x_4 \mapsto a^pb^q \end{array}\right\}$$

$\theta(x_1x_2x_3x_4) = a^ib^jta^kb^lta^mb^na^pb^q = a^{i-j+m+p}b^{j-l+n+q} = 1_d$
Again there are a number of solutions to this, so to investigate the epimorphism, first note that there are 4 classes of subgroups of order 3, namely $\langle\tau_1 = a\rangle$, $\langle\tau_2 = b\rangle$, $\langle\tau_3 = ab\rangle$ and $\langle\tau_4 = a^2b\rangle$. The cosets are given by

	$\langle\tau_1 = a\rangle$	$\langle\tau_2 = b\rangle$	$\langle\tau_3 = ab\rangle$	$\langle\tau_4 = a^2b\rangle$
1	$\{1, a, a^2\}$	$\{1, b, b^2\}$	$\{1, ab, a^2b^2\}$	$\{1, a^2b, ab^2\}$
2	$\{b, ab, a^2b\}$	$\{a, ab, ab^2\}$	$\{a, a^2b, b^2\}$	$\{a, b, a^2b^2\}$
3	$\{b^2, ab^2, a^2b^2\}$	$\{a^2, a^2b, a^2b^2\}$	$\{ab^2, a^2, b\}$	$\{ab, b^2, a^2\}$
4	$\{t, at, a^2t\}$	$\{t, bt, b^2t\}$	$\{t, abt, a^2b^2t\}$	$\{t, a^2bt, ab^2t\}$
5	$\{bt, abt, a^2bt\}$	$\{at, abt, ab^2t\}$	$\{at, a^2bt, b^2t\}$	$\{at, bt, a^2b^2t\}$
6	$\{b^2t, ab^2t, a^2b^2t\}$	$\{a^2t, a^2bt, a^2b^2t\}$	$\{ab^2t, a^2t, bt\}$	$\{abt, b^2t, a^2t\}$

The action of an element of order 2 a^ib^jt will always give 3 cycles of length 2 so they will not give any conical point.

The action of $a^{\pm1}$ and $(ab)^{\pm1}$ will give 6 cycles of length 1, that is 6 conical points on each of $\langle\tau_1 = a\rangle$ and $\langle\tau_2 = ab\rangle$.

For all other elements a^kb^l the action will only give cycles of length 3 and hence no conical points.

So any epimorphism will give two classes of non-conjugate and normal trigonal morphisms and two classes of automorphisms of order 3 with quotient surface of genus 2.

Δ_4: 1) $\Delta(0;[2,2,2,6]) \to C_{18}$; $Order(\theta(x_1x_2x_3)) = 2$ so there can not be any epimorphism.

2) $\Delta(0;[2,2,2,6]) \to C_3 \times C_6$; $Order(\theta(x_1x_2x_3)) = 2$ so there can not be any epimorphism.

3) $\Delta(0;[2,2,2,6]) \to C_3 \times D_3$; $Order(\theta(x_1x_2x_3)) = 2$ so there can not be any epimorphism.

4,5) There are no elements of order 6 in D_9 or $\langle 3,3,3,2\rangle$ so there can not be any epimorphism.

Since the only group Δ that is maximal is the one with signature $s(\Delta_4) = (0;[2,2,2,6])$ the above shows that there can not exist a Riemann surface having an automorphism group of order 18. □

The following remarks are usefull when investigating the higher cases as these epimorphisms will extend to groups of higher order.

Remark 2.2.1. *The epimorphism $\Delta(0;[3,6,6]) \to C_3 \times C_6$ gives one normal trigonal morphism, two automorphisms of order 3 with quotient surface of genus 1 and one automorphism of order 3 with quotient surface of genus 2.*

Remark 2.2.2. *The epimorphisms $\Delta(0;[3,6,6]) \to C_3 \times D_3$ give either two non-conjugated and normal trigonal morphisms, and one class of automorphisms of order 3 with quotient surface of genus 2 or one normal trigonal morphism, one class of automorphisms of order 3 with quotient surface of genus 1 and one normal automorphism of order 3 with quotient surface of genus 2.*

Remark 2.2.3. *The epimorphisms $\Delta(0;[2,2,3,3]) \to C_3 \times D_3$ give either one normal trigonal morphisms, one class of automorphisms of order 3 with quotient surface of genus 1 and one normal automorphism of order 3 with quotient surface of genus 2 or one class of* **non-normal** *trigonal morphisms and two normal automorphisms of order 3 with quotient surface of genus 2.*

Remark 2.2.4. *The epimorphism $\Delta(0;[2,2,3,3]) \to \langle 3,3,3,2\rangle$ gives two classes of non-conjugate and normal trigonal morphisms and two classes of automorphisms of order 3 with quotient surface of genus 2.*

Proposition 2.2.5. *There exists no trigonal Riemann surface of genus 4 with automorphism group of order 24.*

Proof. For groups of order 24 lemma 2.1.1 gives the following signatures

$s(\Delta_1) = (0;[4,4,4]), \Delta_1 \overset{2}{\hookrightarrow} \Delta(0;[2,4,8]) \overset{3}{\hookrightarrow} \Delta(0;[2,3,8])$
$s(\Delta_2) = (0;[3,4,6])$
$s(\Delta_3) = (0;[3,3,12]), \Delta_3 \overset{2}{\hookrightarrow} \Delta(0;[2,3,24])$
$s(\Delta_4) = (0;[2,8,8]), \Delta_4 \overset{6}{\hookrightarrow} \Delta(0;[2,3,8])$
$s(\Delta_5) = (0;[2,6,12]), \Delta_5 \overset{3}{\hookrightarrow} \Delta(0;[2,3,12])$
$s(\Delta_6) = (0;[2,2,2,4])$

There are 15 groups of order 24

1. $C_{24} = \langle u | u^{24} = 1\rangle$
2. $C_2 \times C_{12} = \langle s,t | s^{12} = t^2 = [s,t] = 1\rangle$
3. $C_2 \times C_2 \times C_6 = \langle u,s,t | u^6 = s^2 = t^2 = [u,s] = [u,t] = [s,t] = 1\rangle$
4. $C_2 \times A_4 = \langle u,s,t | u^2 = s^3 = t^3 = (st)^2 = [u,s] = [u,t] = 1\rangle$
5. $C_2 \times D_6 = \langle u,s,t | u^2 = s^6 = t^2 = (st)^2 = [u,s] = [u,t] = 1\rangle$
6. $C_3 \times D_4 = \langle s,t | s^{12} = t^2 = 1, tst = s^7\rangle$
7. $C_3 \times Q = \langle a,t | a^{12} = t^4 = 1, a^6 = t^2, t^3at = a^7\rangle$
8. $C_4 \times D_3 = \langle s,t | s^{12} = t^2 = 1, tst = s^5\rangle$
9. $C_6 \rtimes C_4 = \langle s,t | s^6 = t^4 = 1, t^3st = s^5\rangle$
10. $D_{12} = \langle s,t | s^{12} = t^2 = (st)^2\rangle$
11. $S_4 = \langle s,t | s^4 = t^2 = (ts)^3\rangle$
12. $\langle 2,3,3\rangle = \langle a,t | a^6 = t^4 = (a^{-1}t)^3 = 1, a^3 = t^2\rangle$
13. $\langle 4,6|2,2\rangle = \langle s,t | s^4 = t^6 = (st)^2 = (s^{-1}t)^2\rangle$
14. $\langle -|2,2,3\rangle = \langle a,t | a^4 = t^4 = (at)^6, a^2 = t^2\rangle$
15. $\langle 2,2,6\rangle = \langle a,t | a^{12} = t^4 = 1, t^2 = a^6, t^3at = a^5\rangle$

Δ_2: $\Delta(0;[3,4,6]) \to G$; The groups of order 24 containing elements of order 3, 4 and 6 are

$$C_{24}, C_{12} \times C_2, C_3 \times D_4, C_3 \times Q, C_4 \times D_3, C_6 \rtimes C_4, D_{12}, \langle 2,3,3\rangle, \langle 4,6|2,2\rangle, \langle -|2,2,3\rangle, \langle 2,2,6\rangle$$

1) $\Delta(0;[3,4,6]) \to C_{24}$;The product of elements of order 3, 4 and 6 can never be the identity. Hence there is no epimorphism.

2) $\Delta(0;[3,4,6]) \to C_{12} \times C_2$; The elements of order 3 are $s^{\pm 4}$, the elements of order 4 are $s^{\pm 3}, s^{\pm 3}t$. Their product has order 12. Hence there is no epimorphism.

6) $\Delta(0;[3,4,6]) \to C_3 \times D_4$; As in the previous case the elements of order 3 and order 4 have product of order 12. Hence there is no epimorphism.

7) $\Delta(0;[3,4,6]) \to C_3 \times Q$; The product of elements of order 3 and 6 has order 6 or 2. Hence there is no epimorphism.

8) $\Delta(0;[3,4,6]) \to C_4 \times D_3$; Again the product of elements of order 3 and 6 has order 6 or 2. Hence there is no epimorphism.

9) $\Delta(0;[3,4,6]) \to C_6 \rtimes C_4$; The elements of order 3 are, $s^{\pm 2}$, the elements of order 4 are $t^{\pm 1}s^i$. Their product has order 4. Hence there is no epimorphism.

10) $\Delta(0;[3,4,6]) \to D_{12}$; The product of elements of order 3 and 4 in D_{12} has order 12. Hence there is no epimorphism.

12) $\Delta(0;[3,4,6]) \to \langle 2,3,3\rangle$; There are epimorphisms in this case. There is only one conjugacy class of elements of order 3 represented by a^2.

    Now the elements of order 3 when acting on $\langle a^2\rangle$-cosets give 2 fixed points (that is conical points of order 3), the elements of order 4 give no conical points and the elements of order 6 give cycles of the type $(a_1,a_2,a_3,a_4,a_5,a_6)(a_7,a_8,)$, that is 1 conical point of order 3. Hence, there are 3 conical points in total, and the signature is $s(\theta^{-1}\langle a^2\rangle) = (1;[3,3,3])$ and the surface $\mathbb{H}/ker(\theta)$ is not trigonal. (See example 1.3.1).

13) $\Delta(0;[3,4,6]) \to \langle 4,6|2,2\rangle$; The product of an element of order 3 ($t^{\pm 2}$) and an element of order 4 ($s^{\pm 1}$ or $s^{\pm 1}t^{\pm 2}$) has order 4. Hence, there is no epimorphism.

14) $\Delta(0;[3,4,6]) \to \langle -|2,2,3\rangle$; The product of an element of order 3 and an element of order 6 is either an element of order 2 or an element of order 6. Hence, there is no epimorphism.

15) $\Delta(0;[3,4,6]) \to \langle 2,2,6\rangle$; The product of an element of order 3 and an element of order 4 has order 12 or 4. Hence, there is no epimorphism.

Δ_3: $\Delta(0;[3,3,12]) \to G$; The groups of order 24 containing elements of order 12 are

$$C_{24}, C_{12}\times C_2, C_3\times D_4, C_3\times Q, C_4\times D_3, D_{12}, \langle 2,2,6\rangle$$

1) $\Delta(0;[3,3,12]) \to C_{24}$; The product of two elements of order 3 is either the identity or an element of order 3. Thus, there can not be any epimorphism.

2) $\Delta(0;[3,3,12]) \to C_{12}\times C_2$; $Order(\theta(x_1x_2))$ divides 3 so there is no epimorphism.

6) $\Delta(0;[3,3,12]) \to C_3\times D_4$; $Order(\theta(x_1x_2))$ divides 3 so there is no epimorphism.

7) $\Delta(0;[3,3,12]) \to C_3\times Q$; $Order(\theta(x_1x_2))$ divides 3 so there is no epimorphism.

8) $\Delta(0;[3,3,12]) \to C_4\times D_3$; $Order(\theta(x_1x_2))$ divides 3 so there is no epimorphism.

10) $\Delta(0;[3,3,12]) \to D_{12}$; $Order(\theta(x_1x_2))$ divides 3 so there is no epimorphism.

15) $\Delta(0;[3,3,12]) \to \langle 2,2,6\rangle$; $Order(\theta(x_1x_2))$ divides 3 so there is no epimorphism.

Δ_4: $\Delta(0;[2,8,8]) \to G$; There is only one group of order 24 that contains elements of order 8, namely C_{24}. The elements of order 8 are $u^{\pm 3}$. The product of two such elements is either the identity or an element of order 4. Hence there can not be any epimorphism $\Delta(0;[2,8,8]) \to C_{24}$.

Δ_5: $\Delta(0;[2,6,12]) \to G$; Again, the groups of order 24 containing elements of order 12 are

$$C_{24}, C_{12} \times C_2, C_3 \times D_4, C_3 \times Q, C_4 \times D_3, D_{12}, \langle 2,2,6\rangle$$

1) $\Delta(0;[2,6,12]) \to C_{24}$; $Order(\theta(x_1x_2)) = 3$. Thus, there is no epimorphism.
2) $\Delta(0;[2,6,12]) \to C_{12} \times C_2$; $Order(\theta(x_1x_2)) = 3$ or 6. Thus, there is no epimorphism.
6) $\Delta(0;[2,6,12]) \to C_3 \times D_4$; The elements of order 2 are t, s^6, ts^6, $ts^{\pm 3}$, the elements of order 6 are $s^{\pm 2}, ts^i$ and the elements of order 12 are $s^{\pm 1}$, $s^{\pm 5}$. So there are epimorphisms. The action of elements of order 2 on $\langle s^4\rangle$-cosets gives no conical points. The action of an element of order 3 gives 4 conical points and the action of an element of order 12 gives 2 conical points. Thus in total there are 6 conical points in the Riemann surface uniformized by $\theta^{-1}(\langle s^4\rangle$. The Riemann surface $\mathbb{H}/ker(\theta)$ is trigonal. Since $\langle s^4\rangle$ is central in $C_3 \times D_4$, there is a normal trigonal morphism in $C_3 \times D_4$.
7) $\Delta(0;[2,6,12]) \to C_3 \times Q$; $Order(\theta(x_1x_2)) = 3$. Thus, there is no epimorphism.
8) $\Delta(0;[2,6,12]) \to C_4 \times D_3$; $Order(\theta(x_2x_3)) = 4$ or 12. Thus, there is no epimorphism.
10) $\Delta(0;[2,6,12]) \to D_{12}$; $Order(\theta(x_2x_3)) = 4$ or 12. Thus, there is no epimorphism.
15) $\Delta(0;[2,6,12]) \to \langle 2,2,6\rangle$; $Order(\theta(x_2x_3)) = 4$ or 12. Thus, there is no epimorphism.

Δ_6: $(0;[2,2,2,4])$ can not produce any trigonal surfaces since the orders of the generators are coprime to 3.

The only maximal groups are the groups with signatures $\Delta(0;[3,4,6])$ and $\Delta(0;[2,2,2,4])$, since neither of them give any trigonal surfaces the result follows.

□

Remark 2.2.5. *The epimorphism* $\Delta(0;[2,6,12]) \to C_3 \times D_4$ *gives a unique trigonal morphism in* $C_3 \times D_4$.

Remark 2.2.6. *There is no epimorphism from a group with signature* $(0;[3,3,12])$ *onto a group of order 24.*

Remark 2.2.7. *There is no epimorphism from a group with signature* $(0;[2,8,8])$ *onto a group of order 24.*

Proposition 2.2.6. *The subspace of* $\mathcal{M}_4^3$ *formed by cyclic trigonal Riemann surfaces* X_4 *with automorphism groups of order 36 has dimension 1. The automorphism group is* $D_3 \times D_3$ *and the quotient space* $X_4/(D_3 \times D_3)$ *is uniformized by a Fuchsian group with signature* $(0;[2,2,2,3])$ *and these surfaces admit non-normal trigonal morphisms.*

Proof. Lemma 2.1.1 gives the following signatures:

$$\begin{array}{l} s(\Delta_1) = (0;[3,4,4]), \Delta_1 \overset{2:1}{\hookrightarrow} \Delta(0;[2,4,6]) \\ s(\Delta_2) = (0;[3,3,6]), \Delta_2 \overset{2:1}{\hookrightarrow} \Delta(0;[2,3,12]) \\ s(\Delta_3) = (0;[2,6,6]), \Delta_3 \overset{2:1}{\hookrightarrow} \Delta(0;[2,4,6]) \\ s(\Delta_4) = (0;[2,4,12]) \\ s(\Delta_5) = (0;[2,2,2,3]) \end{array}$$

and there are 14 groups of order 36. They are

1. $C_{36} = \langle u | u^{36} = 1 \rangle$
2. $C_{18} \times C_2 = \langle s, u | u^{18} = s^2 = [s,u] = 1 \rangle$
3. $C_{12} \times C_3 = \langle s, u | u^{12} = s^3 = [s,u] = 1 \rangle$
4. $C_6 \times C_6 = \langle s, u | u^6 = s^6 = [s,u] = 1 \rangle$
5. $(C_2 \times C_2) \rtimes C_9 = \langle a, s, t | a^9 = s^2 = t^2 = (st)^2 = 1,$ $a^8 s a = t, a^8 t a = st \rangle$
6. $A_4 \times C_3 = \langle a, s, t | a^3 = s^3 = t^3 = (st)^2 = [a,s] = [a,t] = 1 \rangle$
7. $C_9 \rtimes C_4 = \langle a, t | a^9 = t^4 = 1, t^3 a t = a^8 \rangle$
8. $D_{18} = \langle a, s | a^{18} = s^2 = (as)^2 \rangle$
9. $(C_3 \times C_3) \rtimes_1 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = 1, t^3 a t = a^2,$ $t^3 b t = b^2 \rangle$
10. $T \times C_3 = \langle a, b, t | a^3 = b^4 = t^3 = [a,t] = [b,t] = 1, b^3 a b = a^2 \rangle$
11. $(C_3 \times C_3) \rtimes_2 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = [a,b] = 1, t^3 a t = b,$ $t^3 b t = a^2 \rangle$
12. $\langle 3,3,3,2 \rangle \times C_2 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 = [a,b] = [a,t] =$ $= [b,t] = [s,t] = 1, sas = a^2, sbs = b^2 \rangle$
13. $D_3 \times C_6 = \langle a, s, t | a^6 = s^2 = t^3 = (sa)^2 = [a,t] = [s,t] = 1 \rangle$
14. $(C_3 \times C_3) \rtimes (C_2 \times C_2) = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 =$ $= (st)^2 = [a,b] = [a,t] = [b,s] = (sa)^2 = (tb)^2 \rangle$

Δ_1: There are only 6 groups that have elements of order 4:

1) $\Delta(0;[3,4,4]) \to C_{36}$; The product of two elements of order 4 is an element of order 2 or the identity. Hence there is no epimorphism.

3) $\Delta(0;[3,4,4]) \to C_{12} \times C_3$; The elements of order 4 are $u^{\pm 3}$ and $order(\theta(x_2 x_3))$ divides 2 and so there is no epimorphism.

7) $\Delta(0;[3,4,4]) \to C_9 \rtimes C_4$; The elements of order 4 are of the type $a^i t^{\pm 1}$ for $i = 0, \dots 8$. The product of 2 such elements becomes $a^i t^{\pm 1} a^j t^{\pm 1}$, which is an element of order 3 if the exponents on the t's are different from each other, that is $a^i t^{\pm 1} a^j t^{\mp 1} = a^i a^{-j} = a^{i-j}$. So $\theta(x_1 x_2 x_3) = 1_d$ if $i - j \equiv \pm 3 \bmod 3$ but then $\theta(\Delta_1) \subsetneq C_9 \rtimes C_4$. Hence there is no epimorphism.

9) $\Delta(0;[3,4,4]) \to (C_3 \times C_3) \rtimes_1 C_4$; The elements of order 2 are $t^{\pm 1}$ and $order(\theta(x_2 x_3))$ divides 2. There is no epimorphism.

10) $\Delta(0;[3,4,4]) \to T \times C_3$; The elements of order 4 are $a^k b^{\pm 1}$ and the elements of order 3 are $a^i t^j$ for $i, j = 0, \dots 3$, $(i,j) \neq (0,0)$. $\theta(x_1 x_2 x_3) = 1_d$ if and only if $j = 0$ but then $t \notin \theta(\Delta_1)$. Hence there is no epimorphism.

11) $\Delta(0;[3,4,4]) \to (C_3 \times C_3) \rtimes_2 C_4$; Consider the epimorphism defined as:

$$\left.\begin{array}{l} x_1 \mapsto a^i b^j \\ x_2 \mapsto a^k b^l t^{\pm 1} \\ x_3 \mapsto a^l b^m t^{\pm 1} \end{array}\right\} \theta(x_1 x_2 x_3) = 1_d$$

There are 2 conjugacy classes of subgroups of order 3 with representatives $\langle a \rangle$ and $\langle ab \rangle$.

The elements of order 4 acting on the $\langle a \rangle$- and $\langle ab \rangle$-cosets give no conical points of order 3. The action of elements of order 3 however, gives 6 conical points on one type of cosets and nothing on the other. So there is one class of non-unique trigonal morphisms and one class of automorphism of order 3 with quotient surface of genus 2.

Δ_2 1) $\Delta(0;[3,3,6]) \to C_{36}$; $Order(\theta(x_1 x_2))$ divides 3 so there is no epimorphism.

2) $\Delta(0;[3,3,6]) \to C_{18} \times C_2$; $Order(\theta(x_1 x_2))$ divides 3, thus there is no epimorphism.

3) $\Delta(0;[3,3,6]) \to C_{12} \times C_3$; $Order(\theta(x_1 x_2))$ divides 3 so there is no epimorphism.

4) $\Delta(0;[3,3,6]) \to C_6 \times C_6$; $Order(\theta(x_1 x_2)) = 3$ hence there is no epimorphism.

5) $\Delta(0;[3,3,6]) \to (C_2 \times C_2) \rtimes C_9$; $Order(\theta(x_1 x_2)) = 3$ so there is no epimorphism.

6) $\Delta(0;[3,3,6]) \to A_4 \times C_3$; Consider the epimorphism:

$$\left.\begin{array}{l} x_1 \mapsto at \\ x_2 \mapsto as \\ x_3 \mapsto ats \end{array}\right\} \theta(x_1 x_2 x_3) = 1_d$$

There are 4 conjugacy classes of subgroups of order 3 with representatives $\langle a\rangle$, $\langle s\rangle$, $\langle as\rangle$ and $\langle a^2s\rangle$.

The action of at gives 3 conical points on the $\langle a^2s\rangle$-cosets, and none on the other.

The action of as gives 3 conical points on the $\langle as\rangle$-cosets, and none on the other.

The action of ats gives 6 conical points of order 3 on the $\langle a\rangle$-cosets and none on the $\langle s\rangle$-cosets, the $\langle as\rangle$-cosets and the $\langle a^2s\rangle$-cosets.

All epimorphisms will give equivalent actions on the cosets.

Hence the epimorphism $\Delta(0;[3,3,6]) \to A_4 \times C_3$ produces class of normal trigonal morphism, two classes of automorphisms of order 3 with quotient surface of genus 1 and one class of automorphisms of order 3 with quotient surface of genus 2.

7) $\Delta(0;[3,3,6]) \to C_9 \rtimes C_4$; Elements of order 3 are $a^it^{\pm 1}$ and so the product of 2 such elements $a^it^{\pm 1}a^jt^{\pm 1}$ is an element of order 6 if $i-j \equiv \pm 3 (\mathrm{mod}\ 3)$. But then $\theta(\Delta_2) \le C_3 \rtimes C_4$. Hence there is no epimorphism.

8) $\Delta(0;[3,3,6]) \to D_{18}$; $Order(\theta(x_1x_2))$ divides 3 so there is no epimorphism.

9) $\Delta(0;[3,3,6]) \to (C_3 \times C_3) \rtimes_1 C_4$; $Order(\theta(x_1x_2)) = 3$ so there is no epimorphism.

10) $\Delta(0;[3,3,6]) \to T \times C_3$; $Order(\theta(x_1x_2)) = 3$ so there is no epimorphism.

11) $\Delta(0;[3,3,6]) \to (C_3 \times C_3) \rtimes_2 C_4$; There are no elements of order 6 in $(C_3 \times C_3) \rtimes_2 C_4$;

12) $\Delta(0;[3,3,6]) \to \langle 3,3,3,2\rangle \times C_2$; $Order(\theta(x_1x_2)) = 3$ so there is no epimorphism.

13) $\Delta(0;[3,3,6]) \to D_3 \times C_6$; $Order(\theta(x_1x_2)) = 3$ so there is no epimorphism.

14) $\Delta(0;[3,3,6]) \to D_3 \rtimes D_3$; $Order(\theta(x_1x_2)) = 3$ so there is no epimorphism.

Δ_3

1) $\Delta(0;[2,6,6]) \to C_{36}$; $Order(\theta(x_2x_3))$ divides 3 so there is no epimorphism.

2) $\Delta(0;[2,6,6]) \to C_{18} \times C_2$; The elements of order 2 and 6 only contain powers of u that are multiples of 3. That is, $\theta(\Delta_3) \le C_6 \times C_2$, so there is no epimorphism.

3) $\Delta(0;[2,6,6]) \to C_{12} \times C_3$; $Order(\theta(x_2x_3))$ divides 3 so there is no epimorphism.

4) $\Delta(0;[2,6,6]) \to C_6 \times C_6$; $Order(\theta(x_2x_3))$ can have order 2 but then $\theta(\Delta_3) \le C_6 \times C_2$ hence there is no epimorphism.

5) $\Delta(0;[2,6,6]) \to (C_2 \times C_2) \rtimes C_9;\ \theta(\Delta_3)) \leq (C_2 \times C_2) \rtimes C_3$ so there is no epimorphism.

6) $\Delta(0;[2,6,6]) \to A_4 \times C_3$; $Order(\theta(x_2x_3)) = 3$ hence there is no epimorphism.

7) $\Delta(0;[2,6,6]) \to C_9 \rtimes C_4$; $Order(\theta(x_2x_3))$ divides 3 so there is no epimorphism.

8) $\Delta(0;[2,6,6]) \to D_{18}$; $Order(\theta(x_2x_3))$ divides 3 so there is no epimorphism.

9) $\Delta(0;[2,6,6]) \to (C_3 \times C_3) \rtimes_1 C_4$; $Order(\theta(x_2x_3)) = 3$ so there is no epimorphism.

10) $\Delta(0;[2,6,6]) \to T \times C_3$; $Order(\theta(x_2x_3)$ is either 12 or 3 so there is no epimorphism.

11) $\Delta(0;[2,6,6]) \to (C_3 \times C_3) \rtimes_2 C_4$; There are no elements of order 6 in $(C_3 \times C_3) \rtimes_2 C_4$ so there is no epimorphism.

12) $\Delta(0;[2,6,6]) \to \langle 3,3,3,2\rangle \times C_2$; $Order(\theta(x_2x_3)) = 3$ so there is no epimorphism.

13) $\Delta(0;[2,6,6]) \to D_3 \times C_6$; If $Order(\theta(x_2x_3)) = 2$, then $\theta(\Delta_3) \subsetneqq D_3 \times C_6$ so there is no epimorphism.

14) $\Delta(0;[2,6,6]) \to D_3 \times D_3$; There is one type of epimorphism in this case:

$$\left.\begin{array}{l} x_1 \mapsto ab^2st \\ x_2 \mapsto at \\ x_3 \mapsto bs \end{array}\right\} \theta(x_1x_2x_3) = 1_d$$

There are 3 conjugacy classes of subgroups of order 3 and the classes are represented by $\langle a\rangle$, $\langle b\rangle$ and $\langle ab\rangle$.

The action on the cosets becomes

		$\langle a\rangle$
ab^2st	$\rightsquigarrow$	$(1,12)(2,10)(3,11)(4,9)(5,7)(6,8)$
at	$\rightsquigarrow$	$(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)$
bs	$\rightsquigarrow$	$(1,5,3,4,2,6)(7,12,8,10,9,11)$

		$\langle b\rangle$
ab^2st	$\rightsquigarrow$	$(1,11)(2,12)(3,10)(4,9)(5,7)(6,8)$
at	$\rightsquigarrow$	$(1,8,3,7,2,9)(4,12,5,10,6,11)$
bs	$\rightsquigarrow$	$(1,4)(2,5)(3,6)(7,10)(8,11)(9,12)$

		$\langle ab\rangle$
ab^2st	$\rightsquigarrow$	$(1,12)(2,10)(3,11)(4,7)(5,8)(6,9)$
at	$\rightsquigarrow$	$(1,8,3,7,2,9)(4,12,5,10,6,11)$
bs	$\rightsquigarrow$	$(1,6,2,4,3,5)(7,11,9,10,8,12)$

The action of elements of order 2 on any of the 3 cosets gives no conical points. The action of elements of order 6 gives 6 conical points on both the $\langle a \rangle$- and $\langle b \rangle$-coset and none on the $\langle ab \rangle$-cosets. Thus, there are two classes of normal trigonal morphisms and one class of automorphisms of order 3 with quotient surface of genus 2.

Δ_4 Again there are only 6 groups of order 36 that contain an element of order 4:

1) $\Delta(0; [2, 4, 12]) \to C_{36}$; $Order(\theta(x_1x_2))$ divides 4 so there can be no epimorphism.
3) $\Delta(0; [2, 4, 12]) \to C_{12} \times C_3$; $Order(\theta(x_1x_2)) = 3$ so there can be no epimorphism.
7) $\Delta(0; [2, 4, 12]) \to C_9 \rtimes C_4$; There are no elements of order 12 so there can be no epimorphism.
9) $\Delta(0; [2, 4, 12]) \to (C_3 \times C_3) \rtimes_1 C_4$; $Order(\theta(x_1x_2)) = 4$ so there can be no epimorphism.
10) $\Delta(0; [2, 4, 12]) \to T \times C_3$; $Order(\theta(x_1x_2)) = 4$ so there can be no epimorphism.
11) $\Delta(0; [2, 4, 12]) \to (C_3 \times C_3) \rtimes_2 C_4$; There are no elements of order 12 in $(C_3 \times C_3) \rtimes_2 C_4$ so there can be no epimorphism.

Δ_5

1) $\Delta(0; [2, 2, 2, 3]) \to C_{36}$; $Order(\theta(x_1x_2x_3)) = 2$. Hence, there can be no epimorphism.
2) $\Delta(0; [2, 2, 2, 3]) \to C_{18} \times C_2$; The elements of order 2 and 3 only contain powers of u that are multiples of 3. That is, $\theta(\Delta_3) \leq C_6 \times C_2$, so there is no epimorphism.
3) $\Delta(0; [2, 2, 2, 3]) \to C_{12} \times C_3$; $Order(\theta(x_1x_2x_3)) = 2$ so there can be no epimorphism.
4) $\Delta(0; [2, 2, 2, 3]) \to C_6 \times C_6$; $Order(\theta(x_1x_2x_3)) = 2$ so there can be no epimorphism.
5) $\Delta(0; [2, 2, 2, 3]) \to (C_2 \times C_2) \rtimes C_9$; $Order(\theta(x_1x_2x_3)) = 2$ so there can be no epimorphism.
6) $\Delta(0; [2, 2, 2, 3]) \to A_4 \times C_3$; There can be no epimorphism since $\theta(\Delta) \leq A_4$ or $\theta(\Delta) \leq C_6 \times C_2$.
7) $\Delta(0; [2, 2, 2, 3]) \to C_9 \rtimes C_4$; $Order(\theta(x_1x_2x_3) = 2$ hence there can be no epimorphism.
8) $\Delta(0; [2, 2, 2, 3]) \to D_{18}$; $order(\theta(x_1x_2x_3))$ divides 2 so there can be no epimorphism.
9) $\Delta(0; [2, 2, 2, 3]) \to (C_3 \times C_3) \rtimes_1 C_4$; $Order(\theta(x_1x_2x_3)) = 2$ so there can be no epimorphism.

10) $\Delta(0;[2,2,2,3]) \to T \times C_3$; $Order(\theta(x_1x_2x_3)) = 2$ so there can be no epimorphism.

11) $\Delta(0;[2,2,2,3]) \to (C_3 \times C_3) \rtimes_2 C_4$; $Order(\theta(x_1x_2x_3)) = 2$ so there can be no epimorphism.

12) $\Delta(0;[2,2,2,3]) \to \langle 3,3,3,2\rangle \times C_2$; $Order(\theta(x_1x_2x_3))$ can be 3 but then still $\theta(\Delta) \lneq \langle 3,3,3,2\rangle \times C_2$ (in fact $\theta(\Delta) = D_6$) so there can be no epimorphism.

13) $\Delta(0;[2,2,2,3]) \to D_3 \times C_6$; If $Order(\theta(x_1x_2x_3)) = 3$ then $\theta(\Delta_5) = D_6$, so there can be no epimorhism.

14) $\Delta(0;[2,2,2,3]) \to D_3 \times D_3$; There is one type of epimorphism:

$$\left.\begin{array}{l} x_1 \mapsto s \\ x_2 \mapsto tb \\ x_3 \mapsto sta \\ x_4 \mapsto a^2b \end{array}\right\} \theta(x_1x_2x_3x_4) = 1_d$$

The elements of order 2 do not give any conical point while the action of an element of order 3 gives

		$\langle a \rangle$
a^2b	$\rightsquigarrow$	$(1,2,3)(4,5,6)(7,8,9)(10,12,11)$

		$\langle b \rangle$
a^2b	$\rightsquigarrow$	$(1,3,2)(4,5,6)(7,8,9))(10,11,12)$

		$\langle ab \rangle$
a^2b	$\rightsquigarrow$	$(1,2,3)(4)(6)(5)(7)(8)(9)(10,11,12)$

Thus, the epimorphism $\Delta(0;[2,2,2,3]) \to D_3 \times D_3$ produces one class of non-normal trigonal morphisms and two classes of normal automorphism of order 3 with quotient surface of genus 2.
Hence, the only fuchsian group that will give a trigonal Riemann surface with automorphism group of order 36 is the group with signature (0;[2,2,2,3]) and the automorphsim group is $D_3 \times D_3$.
The dimension formula for Teichmüller spaces (thm 1.5.1) gives that the subspace of $\mathcal{M}_4^3$ formed by these Riemann surfaces has dimension 1.

□

Remark 2.2.8. *The epimorphism* $\Delta(0;[3,4,4]) \to (C_3 \times C_3) \rtimes_2 C_4$ *produces one class of non-normal trigonal morphisms and one class of automorphisms of order 3 with quotient surface of genus 2.*

Remark 2.2.9. *The epimorphism* $\Delta(0;[3,3,6]) \to A_4 \times C_3$ *produces one class of normal trigonal morphism and two classes of automorphisms of order 3 with quotient surface of genus 1 and one class of automorphisms of order 3 with quotient surface of genus 2.*

Remark 2.2.10. *The epimorphism $\Delta(0;[2,6,6]) \to D_3 \times D_3$ produces two classes of normal trigonal morphisms and one class of automorphisms of order 3 with quotient surface of genus 2.*

Proposition 2.2.7. *There is no Riemann surface of genus 4 with automorphism group of order 42.*

Proof. Lemma 2.1.1 gives the following signature:

$$s(\Delta) = (0;[3,3,21]), \Delta \overset{2}{\hookrightarrow} \Delta(0;[2,3,42])$$

and there are 2 groups of order 21

$$G \cong C_{21} = \langle u | u^{21} = 1 \rangle$$
$$G \cong C_7 \rtimes C_3 = \langle s,t | s^3 = t^7 = 1, s^2ts = t^4 \rangle$$

1) $\Delta(0;[3,3,21]) \to C_{21}$; The product of 2 elements of order 3 in C_{21} is either the identity or an element of order 3. Hence there is no epimorphism.

2) $\Delta(0;[3,3,21]) \to C_7 \rtimes C_3$ There are no elements of order 21 in $C_7 \rtimes C_3$ and so there can not be any epimorphism.

Since the signature $(0;[3,3,21])$ does not yield any epimorphism there can not be any epimorphism from a group with signature $(0;[2,3,42])$ onto a group of order 42. □

Proposition 2.2.8. *There is no Riemann surface of genus 4 with automorphism group of order 48.*

Proof. From remark 2.2.6 we know there is no epimorphism from $\Delta(0;[3,3,12])$ into a group of order 24 and hence there can be no epimorphism from $\Delta(0;[2,3,24])$ into a group of order 48.

□

Proposition 2.2.9. *There is no Riemann surface of genus 4 with automorphism group of order 54.*

Proof. Lemma 2.1.1 gives the following signature

$$s(\Delta) = (0;[3,3,9]), \Delta \overset{2}{\hookrightarrow} (0;[2,3,18])$$
$$\Delta \overset{4}{\hookrightarrow} (0;[2,3,9])$$

There are 5 groups of order 27

$$G \cong C_{27} = \langle u | u^{27} = 1 \rangle$$
$$G \cong C_9 \times C_3 = \langle s,t | s^9 = t^3 = [s,t] = 1 \rangle$$
$$G \cong C_3 \times C_3 \times C_3 = \langle u,s,t | u^3 = s^3 = t^3 = [u,s] = [s,t] = [u,t] = 1 \langle$$
$$G \cong C_9 \rtimes C_3 = \langle s,t | s^9 = t^3 = 1, s^2st = s^7 \rangle$$
$$G \cong (C_3 \times C_3) \rtimes C_3 = \langle u,s,t | u^3 = s^3 = t^3 = [u,s] = [u,t] = 1, tst = su \rangle$$

where $C_3 \times C_3 \times C_3$ and $(C_3 \times C_3) \rtimes C_3$ are not interesting because they do not contain elements of order 9.

1) $\Delta(0;[3,3,9]) \to C_{27}$; The elements of order 3 are $u^{\pm 7}$ so $Order(\theta(x_1x_2))$ divides 3. Hence there can be no epimorphism.

2) $\Delta(0;[3,3,9]) \to C_9 \times C_3$; The elements of order 3 are $t^i s^{\pm 3}$ and so $Order(\theta(x_1x_2))$ divides 3 and again there can be no epimorphism.

3) $\Delta(0;[3,3,9]) \to C_9 \rtimes C_3$; The elements of order 3 are $s^{\pm 3}, t^{\pm 1}, t^{\pm 1}s^{\pm 3}$, hence $Order(\theta(x_1x_2))$ divides 3 and so there can be no epimorphism.

Since there are no epimorphisms from a group with signature $(0;[3,3,9])$ into a group of order 27 there are no epimorphsims from $\Delta(0;[2,3,18])$ onto a group of order 54. □

Proposition 2.2.10. *There is no Riemann surface with automorphism group of order 60.*

Proof. Lemma 2.1.1 gives the signature

$$s(\Delta) = (0;[2,3,15])$$

Now there are 13 groups of order 60 and they are

1. $A_5 = \langle a,b | a^2 = b^3 = (ab)^5 \rangle$
2. $C_{30} \times C_2 = \langle s,t | s^2 = t^{30} = [s,t] = 1 \rangle$
3. $C_5 \times A_4 = \langle s,t,u | s^3 = t^3 = (st)^2 = u^5 = [s,u] = [t,u] = 1 \rangle$
4. $C_{60} = \langle u | u^{60} = 1 \rangle$
5. $C_{15} \rtimes_{14} C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^{14} \rangle$
6. $C_{15} \rtimes_2 C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^2 \rangle$
7. $C_{15} \rtimes_4 C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^4 \rangle$
8. $C_{15} \rtimes_{11} C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^{11} \rangle$
9. $C_{15} \rtimes_{13} C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^{13} \rangle$
10. $D_{30} = \langle s,t | s^{15} = t^2 = (st)^2 \rangle$
11. $D_3 \times D_5 = \langle a,b,s,t | a^3 = b^2 = s^5 = t^2 = (ab)^2 = (st)^2 =$
    $= [a,s] = [b,s] = [a,t] = [b,t] = 1 \rangle$
12. $D_3 \times C_{10} = \langle s,t,u | s^3 = t^2 = u^{10} = (st)^2 = [s,u] = [t,u] = 1 \rangle$
13. $C_6 \times D_5 = \langle s,t,u | s^5 = t^2 = u^6 = (st)^2 = [s,u] = [t,u] = 1 \rangle$

There is no element of order 15 in A_5. In all the other cases $Order(\theta(x_1x_2))$ divides 6 so there can be no epimorphisms from a group with signature $(0;[2,3,15])$ onto a group of order 60.
This gives the desired result.

□

Proposition 2.2.11. *(i) There is a unique cyclic trigonal Riemann surface Y_4 with automorphsim group of order 72 and non-normal trigonal morphism. Its automorphism group is $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$.*

(ii) *There is another cyclic trigonal Riemann surface X_4 with automorphism group of order 72 and normal trigonal morphism. Its automorphism group is $Aut(X_4) = S_4 \times C_3$.*

Proof. The maximal signatures, when the automorphism group of the surface has order 72, are

$$s(\Delta_1) = (0; [2, 4, 6])$$
$$s(\Delta_2) = (0; [2, 3, 12])$$

Δ_1: By the remarks 2.2.8 and 2.2.10 the only groups to be checked are the extensions of $(C_3 \times C_3) \rtimes_2 C_4$ and $D_3 \times D_3$ containing 2 conjugacy classes of elements of order 3. There is only one such group $(C_3 \times C_3) \rtimes D_4$ which has presentation

$$\langle a, b, s, t | a^3 = b^3 = t^4 = s^2 = [a, b] = [s, b] = (st)^2 = (sa)^2 = 1, t^3 at = b, t^3 bt = a^2 \rangle$$

The epimorphism $\theta : (0; [2, 4, 6]) \to (C_3 \times C_3) \rtimes_2 D_4$ defined by

$$\left.\begin{array}{l} x_1 \mapsto s \\ x_2 \mapsto ta \\ x_3 \mapsto stb \end{array}\right\} \theta(x_1 x_2 x_3) = 1_d$$

yields a Riemann surface $\mathbb{H}/ker(\theta)$ with non-normal trigonal morphism. (In fact the trigonal morphisms are induced by $\langle ab \rangle$ and $\langle a^2 b \rangle$).

Δ_2: By the remarks 2.2.5 and 2.2.9, the only groups to be checked in this case are the extensions of $A_4 \times C_3$ and $C_3 \times D_4$ containing elements of order 12. There is only one such group

$$S_4 \times C_3 = \langle t, a, b | t^2 = a^3 = b^3 = (ta)^4 = [t, b] = [a, b] = 1 \rangle,$$

where b is a central element. Consider the epimorphism:

$$\left.\begin{array}{l} x_1 \mapsto t \\ x_2 \mapsto ab \\ x_3 \mapsto a^2 t b^2 \end{array}\right\} \theta(x_1 x_2 x_3) = 1_d$$

The action of $\theta(x_3)$ on the $\langle b \rangle$-cosets leaves $24/4$ conic points of order $12/4 = 3$. Thus b is a normal trigonal morphism. By González's theorem (thm 1.4.2) and remark 2.2.9 it is the unique trigonal morphism.

Thus, the fuchsian group with signature $(0; [2, 4, 6])$ gives a trigonal Riemann surface Y_4 with automorphism group $(C_3 \times C_3) \rtimes D_4$ of order 72. Y_4 has non-normal trigonal morphism.
The fuchsian group with signature $(0; [2, 3, 12])$ gives a trigonal Riemann surface X_4 with automorphism group $S_4 \times C_3$ of order 72. X_4 has normal (unique) trigonal morphism.

□

Proposition 2.2.12. *There is no Riemann surface of genus 4 with automorphism group of order 90.*

Proof. For $|G| = 15$ lemma 2.1.1 gives the following signature:

$$s(\Delta) = (0; [5, 5, 5]), \Delta \overset{2}{\hookrightarrow} \Delta(0; [2, 5, 10]) \overset{3}{\hookrightarrow} \Delta(0; [2, 3, 10])$$

and there is only one group of order 15 namely

$$G \cong C_{15} = \langle u | u^{15} \rangle$$

$\Delta(0; [5, 5, 5]) \to C_{15}$; The elements of order 5 do not generate C_{15}. Hence, there is no possible epimorphism.
Since the signature $(0; [5, 5, 5])$ does not yield any epimorphism there can not exist any epimorphism from a group with signature $(0; [2, 3, 10])$ onto a group of order 90.

□

Proposition 2.2.13. *There is no Riemann surface of genus 4 with automorphism group of order 108.*

Proof. By the proof of proposition 2.2.9 there are no epimorphisms from a group with signature $(0; [3, 3, 9])$ into a group of order 27. Hence there can be no epimorphisms from $\Delta(0; [2, 3, 9])$ onto a group of order 108. □

Proposition 2.2.14. *There is no cyclic trigonal Riemann surface with automorphism group of order 120.*

Proof. Lemma 2.1.1 gives the only signature

$$(0; [2, 4, 5])$$

however all the generators have order coprime to 3, therefore the signature can not produce any trigonal morphism. □

Proposition 2.2.15. *There is no Riemann surface of genus 4 with automorphism group of order 144.*

Proof. From remark 2.2.7 we know there is no epimorphism from $\Delta(0; [2, 8, 8])$ into a group of order 24 and hence there can be no epimorphism from $\Delta(0; [2, 3, 8])$ into a group of order 144. □

Proposition 2.2.16. *There is no Hurwitz surface of genus 4.*

This is in fact a well known fact but the proof of this in this case is very short and elegant:

Proof. A Riemann surface X_4 is a Hurwitz surface if $X_4/Aut(X_4)$ is uniformized by a Fuchsian group with signature $(0; [2, 3, 7])$ and $|Aut(X_4)| = 252$. There are several groups of order 252 (46 of them). We prove that there exists no epimorphism from the non-normal subgroup $\Delta(0; [2, 7, 7])$ onto a group of order 28. Notice that $\Delta(0; [2, 7, 7])$ has index 9 in $\Delta(0; [2, 3, 7])$
There are only 4 groups of order 28, namely

1. $C_{28} = \langle u | u^{28} = 1 \rangle$
2. $C_{14} \times C_2 = \langle s, t | s^{14} = t^2 = [s, t] = 1 \rangle$
3. $D_{14} = \langle s, t | s^{14} = t^2 = (st)^2 = 1 \rangle$
4. $\langle 2, 2, 7 \rangle = s, t | s^{14} = t^4 = 1, s^7 = t^2, t^3 st = s^{-1} \rangle$

1. $\Delta(0; [2, 7, 7]) \to C_{28}$; The elements of order 7 are $u^{\pm 4}, u^{\pm 8}, u^{\pm 12}$ and so $Order(\theta(x_2 x_3))$ is either 4 or 1. In any case there can be no epimorphism.

2. $\Delta(0; [2, 7, 7]) \to C_{14} \times C_2$; The elements of order 7 are $s^{\pm 2}, s^{\pm 4}, s^{\pm 6}$ and so $Order(\theta(x_2 x_3))$ is either 7 or 1. Hence there can be no epimorphism.

3. $\Delta(0; [2, 7, 7]) \to D_{14}$; The elements of order 7 are $s^{\pm 2}, s^{\pm 4}, s^{\pm 6}$ and so $Order(\theta(x_2 x_3))$ is either 7 or 1. Hence there can be no epimorphism.

4. $\Delta(0; [2, 7, 7]) \to \langle 2, 2, 7 \rangle$; Elements of order 7 are $s^{\pm 2}, s^{\pm 4}, s^{\pm 6}$ and so $Order(\theta(x_2 x_3))$ is either 7 or 1. Hence there can be no epimorphism.

Since there is no epimorphism from a group with signature $(0; [2, 7, 7])$ into a group of order 28 the signature $(0; [2, 3, 7])$ can not produce a surface of genus 4 with automorphism group of maximal order 252=84(4-1). □

2.3 Main results

Now the propositions (2.2.1) - (2.2.16) together give the main theorems:

Theorem 1. *(i) There is a uniparametric family of cyclic trigonal Riemann surfaces $X_4(\lambda)$ of genus 4 with non-normal trigonal morphisms. $Aut(X_4(\lambda)) = D_3 \times D_3$ and $X_4(\lambda)/Aut(X_4(\lambda))$ are uniformized by the Fuchsian group Δ with signature $s(\Delta) = (0; [2, 2, 2, 3])$.*

(ii) There is one cyclic trigonal Riemann surface Y_4 of genus 4 with non-normal trigonal morphisms. $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$ and $Y_4/Aut(Y_4))$ is the sphere with 3 conic points of order 2, 4 and 6 respectively.

Theorem 2. *The space $\mathcal{M}_4^3$ of cyclic trigonal Riemann surfaces of genus 4 form a disconnected subspace of the moduli space $\mathcal{M}_4$ of dimension 3.*

1. *The subspace of $\mathcal{M}_4^3$ formed by Riemann surfaces of genus 4 with automorphism group of order 6 has dimension 2 in $\mathcal{M}_4^3$. The automorphism group of the Riemann surfaces is either C_6 or D_3.*

2. *The subspace of $\mathcal{M}_4^3$ formed by Riemann surfaces of genus 4 with automorphism group of order 12 has dimension 1 in $\mathcal{M}_4^3$. The automorphism group of the Riemann surfaces is either $C_2 \times C_6$ or D_6.*

3. *The subspace of* $\mathcal{M}_4^3$ *formed by Riemann surfaces* $X_4(\Delta)$ *of genus 4 with automorphism group of order 36 has dimension 1 in* $\mathcal{M}_4^3$*. The automorphism group of the Riemann surfaces is* $D_3 \times D_3$ *and the surfaces admit non-normal trigonal morphisms.*

4. *There are exactly 2 cyclic trigonal Riemann surfaces* X_4 *and* Y_4 *of genus 4 with automorphism groups of order 72.*

   (i) X_4 *has a normal trigonal morphism and* $Aut(X_4) = S_4 \times C_3$.

   (ii) Y_4 *has non-normal trigonal morphisms and* $Aut(Y_4) = (C_3 \times C_3) \rtimes D_4$

Appendix A

List of groups

We follow Coxeter's and Moser's notation for groups up to order 27

Groups of order 12

1. $C_{12} = \langle u | u^{12} = 1 \rangle$
2. $C_6 \times C_2 = \langle s, t | s^6 = t^2 = [s, t] = 1 \rangle$
3. $D_6 = \langle s, t | s^6 = t^2 = (st)^2 = 1 \rangle$
4. $A_4 = \langle s, t | s^3 = t^3 = (st)^2 = 1 \rangle$

Elements

Order	Elements	Number of elements
2	st, ts^2t, s^2t^2	3
3	$s^{\pm 1}$, $t^{\pm 1}$, $(st^2)^{\pm 1}$, $(s^2t)^{\pm 1}$	8
Total number of elements		12

5. $T = \langle s, t | t^3 = s^4 = 1, s^3ts = t^2 \rangle$ $(= \langle 2, 2, 3 \rangle)$

Elements

Order	Elements	Number of elements
2	s^2	1
3	$t^{\pm 1}$	2
4	$s^{\pm 1}$, $ts^{\pm 1}$, $t^2s^{\pm 1}$	6
6	ts^2, t^2s^2	2
Total number of elements		12

Groups of order 15

$C_{15} = \langle u | u^{15} = 1 \rangle$

Groups of order 18

1. $C_{18} = \langle u | u^{18} = 1 \rangle$
2. $C_3 \times C_6 = \langle s, t | s^3 = t^6 = [s, t] = 1 \rangle$
3. $C_3 \times D_3 = \langle s, t | s^3 = t^6 = 1, t^5 st = s^2 \rangle$

Elements

Order	Elements	Number of elements
2	$s^i t^3$ $(i = 0, \ldots, 2)$	3
3	$s^{\pm 1}$, $t^{\pm 2}$, $s^{\pm 1} t^{\pm 2}$	8
6	$t^{\pm 1}$, $s^{\pm 1} t^{\pm 1}$	6
Total number of elements		18

4. $D_9 = \langle s, t | s^2 = t^9 = (st)^2 = 1 \rangle$

5. $\langle 3, 3, 3, 2 \rangle = \langle a, b, t | a^3 = b^3 = t^2 = [a, b] = 1, tat = a^{-1}, tbt = b^{-1} \rangle$

Elements

Order	Elements	Number of elements
2	$a^i b^j t$	9
3	$a^i b^j$, $(i, j) \neq (0, 0)$	8
Total number of elements		18

Groups of order 21

1. $C_{21} = \langle u | u^{21} = 1 \rangle$
2. $C_7 \rtimes C_3 = \langle s, t | s^3 = t^7 = 1, s^2 ts = t^4 \rangle$

Elements

Order	Elements	Number of elements
3	$s^{\pm 1} t^i$, $(i = 0, \ldots, 6)$	14
7	b^i, $(i = 1, \ldots, 6)$	6
Total number of elements		21

Groups of order 24

1. $C_{24} = \langle u | u^{24} = 1 \rangle$

2. $C_2 \times C_{12} = \langle s, t | s^{12} = t^2 = [s, t] = 1 \rangle$

3. $C_2 \times C_2 \times C_6 = \langle u, s, t | u^6 = s^2 = t^2 = [u, s] = [u, t] = [s, t] = 1 \rangle$

4. $C_2 \times A_4 = \langle u, s, t | u^2 = s^3 = t^3 = (st)^2 = [u, s] = [u, t] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	st, ts^2t s^2t^2 u ust, uts^2t us^2t^2	7
3	$s^{\pm 1}$, $t^{\pm 1}$, st^2 s^2t, ts^2, t^2s	8
6	$us^{\pm 1}$, $ut^{\pm 1}$, ust^2 us^2t, uts^2, ut^2s	8
Total number of elements		24

5. $C_2 \times D_6 = \langle u, s, t | u^2 = s^6 = t^2 = (st)^2 = [u, s] = [u, t] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	s^3, s^it, u, us^3 us^it $(i = 0, \ldots, 5)$	15
3	$s^{\pm 2}$	2
6	$s^{\pm 1}$, $us^{\pm 2}$, $us^{\pm 1}$	6
Total number of elements		24

6. $C_3 \times D_4 = \langle s, t | s^{12} = t^2 = 1, tst = s^7 \rangle$

Elements

Order	Elements	Number of elements
2	t, s^6, ts^6, $ts^{\pm 3}$	5
3	$s^{\pm 4}$	2
4	$s^{\pm 3}$	2
6	$s^{\pm 2}$, $ts^{\pm 1}$, $ts^{\pm 2}$, $ts^{\pm 4}$, $ts^{\pm 5}$	10
12	$s^{\pm 1}$, $s^{\pm 5}$	4
Total number of elements		24

7. $C_3 \times Q = \langle a, t | a^{12} = t^4 = 1, a^6 = t^2, t^3at = a^7 \rangle$

Elements

Order	Elements	Number of elements
2	t^2	1
3	$a^{\pm 4}$	2
4	$a^{\pm 3}$, $t^{\pm 1}$, $t^{\pm 1}a^3$	6
6	$a^{\pm 2}$	2
12	$a^{\pm 1}$, $a^{\pm 5}$, $t^{\pm 1}a$ $t^{\pm 1}a^2$, $t^{\pm 1}a^4$, $t^{\pm 1}a^5$	12
Total number of elements		24

8. $C_4 \times D_3 = \langle s,t | s^{12} = t^2 = 1, tst = s^5 \rangle$

Elements

Order	Elements	Number of elements
2	t, s^6, ts^6, $ts^{\pm 2}$, $ts^{\pm 4}$	7
3	$s^{\pm 4}$	2
4	$s^{\pm 3}$, $ts^{\pm 1}$, $ts^{\pm 3}$, $ts^{\pm 5}$	8
6	$s^{\pm 2}$	2
12	$s^{\pm 1}$, $s^{\pm 5}$	4
Total number of elements		24

9. $C_6 \rtimes C_4 = \langle s,t | s^6 = t^4 = 1, t^3st = s^5 \rangle$

Elements

Order	Elements	Number of elements
2	s^3, t^2, t^2s^3	3
3	$s^{\pm 2}$	2
4	$t^{\pm 1}s^i$, $(s = 0, \ldots, 5)$	12
6	$s^{\pm 1}$, $t^2s^{\pm 1}$, $t^2s^{\pm 2}$	6
Total number of elements		24

10. $D_{12} = \langle s,t | s^{12} = t^2 = (st)^2 \rangle$

11. $S_4 = \langle s,t | s^4 = t^2 = (ts)^3 \rangle$

Elements

Order	Elements	Number of elements
2	$t, s^2, s^2ts^2, s^3ts, ts^2t$ $ts^2ts, ts^2ts^2, ts^2ts^3$	9
3	$ts, st, ts^3, s^2ts, s^3ts^2$ sts^2, s^2ts^3, s^3t	8
4	$ts^2, s^2t, s^{\pm 1}, s^3ts^3, sts$	6
Total number of elements		24

12. $\langle 2,3,3 \rangle = \langle a,t | a^6 = t^4 = (a^{-1}t)^3 = 1, a^3 = t^2 \rangle = Q \rtimes C_3$

Elements

Order	Elements	Number of elements
2	a^3	1
3	$a^{\pm 2}, ta^4, a^4t, ta^5, a^5t, tat, ta^2ta^3$	8
4	$t^{\pm 1}, a^2ta, a^5ta, ata^2, a^4ta^3$	6
6	$a^{\pm 1}, ta, ta^2, at, a^2t, ta^2t, tata^3$	8
Total number of elements		24

13. $\langle 4,6|2,2\rangle = \langle s,t|s^4 = t^6 = (st)^2 = (s^{-1}t)^2\rangle$

Elements

Order	Elements	Number of elements
2	$s^2, t^3, s^2t^3, s^{\pm1}t^{\pm1}, s^{\pm1}t^3$	9
3	$t^{\pm2}$	2
4	$s^{\pm1}, s^{\pm1}t^{\pm2}$	6
6	$t^{\pm1}, s^2t^{\pm1}, s^2t^{\pm2}$	6
Total number of elements		24

14. $\langle -|2,2,3\rangle = \langle a,t|a^4 = t^4 = (at)^6, a^2 = t^2\rangle$

Elements

Order	Elements	Number of elements
2	$a^2, tatata, tatata^3$	3
3	$tata, atat$	2
4	$a^{\pm1}, t^{\pm1}, tat, tatat, ata, atata$ $tata^2, tatata^2, ata^3, atata^3$	12
6	$at, ta, ta^3, ata^2, tata^3, atata^2$	6
Total number of elements		24

15. $\langle 2,2,6\rangle = \langle a,t|a^{12} = t^4 = 1, t^2 = a^6, t^3at = a^5\rangle$

Elements

Order	Elements	Number of elements
2	$t^2, t^{\pm1}a, t^{\pm1}a^3, t^{\pm1}a^5$	7
3	$a^{\pm4}$	2
4	$a^{\pm3}, t^{\pm1}, t^{\pm1}a^2, t^{\pm1}a^4$	8
6	$a^{\pm2}$	2
12	$a^{\pm1}, a^{\pm5}$	4
Total number of elements		24

Groups of order 27

1. $C_{27} = \langle u|u^{27} = 1\rangle$
2. $C_9 \times C_3 = \langle s,t|s^9 = t^3 = [s,t] = 1\rangle$
3. $C_3 \times C_3 \times C_3 = \langle u,s,t|u^3 = s^3 = t^3 = [u,s] = [s,t] = [u,t] = 1\rangle$
4. $C_9 \rtimes C_3 = \langle s,t|s^9 = t^3 = 1, s^2st = s^7\rangle$

Elements

Order	Elements	Number of elements
3	$t^{\pm1}, s^{\pm3}, t^{\pm1}s^{\pm3}$	8
9	$s^{\pm1}, s^{\pm2}, s^{\pm4}$ $t^{\pm1}s^{\pm1}, t^{\pm1}s^{\pm2}, t^{\pm1}s^{\pm4}$	18
Total number of elements		27

5. $(C_3 \times C_3) \rtimes C_3 = \langle u, s, t | u^3 = s^3 = t^3 = [u, s] = [u, t] = 1, tst = su \rangle$

Elements

Order	Elements	Number of elements
3	$u^i s^j t^k$	26
Total number of elements		27

Groups of order 28

1. $C_{28} = \langle u | u^{28} = 1 \rangle$
2. $C_{14} \times C_2 = \langle s, t | s^{14} = t^2 = [s, t] = 1 \rangle$
3. $D_{14} = \langle s, t | s^{14} = t^2 = (st)^2 = 1 \rangle$
4. $\langle 2, 2, 7 \rangle = s, t | s^{14} = t^4 = 1, s^7 = t^2, t^3 st = s^{-1} \rangle$

Elements

Order	Elements	Number of elements
2	$t^2 = s^7$	1
4	ts^i, $(i = 0, \ldots, 13)$	14
7	$s^{\pm 2}, s^{\pm 4}, s^{\pm 6}$	6
14	$s^{\pm 1}, s^{\pm 3}, s^{\pm 5}$	6
Total number of elements		27

Groups of order 36

1. $C_{36} = \langle u | u^{36} = 1 \rangle$
2. $C_{18} \times C_2 = \langle s, u | s^2 = u^{18} = [s, u] = 1 \rangle$
3. $C_{12} \times C_3 = \langle s, u | s^3 = u^{12} = [s, u] = 1 \rangle$
4. $C_6 \times C_6 = \langle s, u | s^6 = u^6 = [s, u] = 1 \rangle$
5. $(C_2 \times C_2) \rtimes C_9 = \langle a, s, t | a^9 = s^2 = t^2 = [s, t] = 1, a^8 sa = t, a^8 ta = st \rangle$

Elements

Order	Elements	Number of elements
2	s, t, st	3
3	$a^{\pm 3}$	2
6	$s^i t^j a^{\pm 3}$, $(i, j) \neq (0, 0)$	6
9	$s^i t^j a^{\pm 1}, s^i t^j a^{\pm 2}, s^i t^j a^{\pm 4}$	24
Total number of elements		36

6. $A_4 \times C_3 = \langle a, s, t | a^3 = s^3 = t^3 = (st)^2 = [a, s] = [a, t] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	st, ts^2t s^2t^2	3
3	$a^{\pm1}$, $s^{\pm1}a^i$, $t^{\pm1}a^i$ $(st^2)^{\pm1}a^i$, $(s^2t)^{\pm1}a^i$	26
6	$sta^{\pm1}$, $ts^2ta^{\pm1}$ $s^2t^2a^{\pm1}$	6
Total number of elements		36

7. $C_9 \rtimes C_4 = \langle a, t | a^9 = t^4 = 1, t^3at = a^8 \rangle$

Elements

Order	Elements	Number of elements
2	t^2	1
3	$a^{\pm3}$	2
4	$a^it^{\pm1}$, $(i = 0, \ldots, 8)$	18
6	$a^{\pm3}t^2$	2
9	$a^{\pm1}, a^{\pm2}, a^{\pm4}$	4
18	$a^{\pm1}t^2, a^{\pm2}t^2, a^{\pm4}t^2$	6
Total number of elements		36

8. $D_{18} = \langle a, s | a^{18} = s^2 = (sa)^2 \rangle$

9. $(C_3 \times C_3) \rtimes_1 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = [a, b] = 1, t^3at = a^2, t^3bt = b^2 \rangle$

Elements

Order	Elements	Number of elements
2	t^2	1
3	a^ib^j, $(i, j) \neq (0, 0)$	8
4	$t^{\pm1}$	2
6	$a^ib^jt^2$, $(i, j) \neq (0, 0)$	8
18	$a^ib^jt^{\pm1}$, $(i, j) \neq (0, 0)$	16
Total number of elements		36

10. $T \times C_3 = \langle a, b, t | a^3 = b^4 = t^3 = [a, t] = [b, t] = 1, b^3ab = a^2 \rangle$

Elements

Order	Elements	Number of elements
2	b^2	1
3	a^it^j, $(i, j) \neq (0, 0)$	8
4	$a^ib^{\pm1}$	6
6	$a^{\pm1}b^2, t^{\pm1}b^2$	4
18	$a^ib^{\pm1}t^j$, $(i, j) \neq (0, 0)$	16
Total number of elements		36

11. $(C_3 \times C_3) \rtimes_2 C_4 = \langle a, b, t | a^3 = b^3 = t^4 = [a, b] = 1, t^3 at = b, t^3 bt = a^2 \rangle$

Elements

Order	Elements	Number of elements
2	$a^i b^j t^2$	10
3	$a^i b^j$, $(i, j) \neq (0, 0)$	8
4	$a^i b^j t^{\pm 1}$	18
Total number of elements		36

12. $\langle 3, 3, 3, 2 \rangle \times C_2 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 = [a, b] = [a, t] = [b, t] = [s, t] = 1, sas = a^2, sbs = b^2 \rangle$

Elements

Order	Elements	Number of elements
2	$a^i b^j s, a^i b^j st, t$	19
3	$a^i b^j$, $(i, j) \neq (0, 0)$	8
4	$a^i b^j t$	8
Total number of elements		36

13. $D_3 \times C_6 = \langle a, s, t | a^6 = s^2 = t^3 = (sa)^2 = [a, t] = [s, t] = 1 \rangle$

Elements

Order	Elements	Number of elements
2	$a^3, a^i s$, $(i = 0; \ldots, 5)$	19
3	$a^2 t^j, t^{\pm 1}$, $(i = 0; \ldots, 2)$	8
6	$a^{\pm 1}, a^i st^{\pm 1}$, $(i = 0; \ldots, 5)$ $a^{\pm 1} t^{\pm 1}, a^3 t^{\pm 1}$	20
Total number of elements		36

14. $D_3 \times D_3 = \langle a, b, s, t | a^3 = b^3 = s^2 = t^2 = (st)^2 = [a, b] = [a, t] = [b, s] = (sa)^2 = (tb)^2 \rangle$

Elements

Order	Elements	Number of elements
2	$a^i s, b^i t, a^i b^j st$	15
3	$a^i b^j$	8
6	$a^i b^{\pm 1} s, a^{\pm} b^j t$	12
Total number of elements		36

Groups of order 60

1. $A_5 = \langle a, b | a^2 = b^3 = (ab)^5 \rangle$
2. $C_{30} \times C_2 = \langle s, t | s^2 = t^{30} = [s, t] = 1 \rangle$

3. $C_5 \times A_4 = \langle s,t,u | s^3 = t^3 = (st)^2 = u^5 = [s,u] = [t,u] = 1 \rangle$
4. $C_{60} = \langle u | u^{60} = 1 \rangle$
5. $C_{15} \rtimes_{14} C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^{14} \rangle$
6. $C_{15} \rtimes_2 C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^2 \rangle$
7. $C_{15} \rtimes_4 C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^4 \rangle$
8. $C_{15} \rtimes_{11} C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^{11} \rangle$
9. $C_{15} \rtimes_{13} C_4 = \langle a,t | a^{15} = t^4 = 1, t^3at = a^{13} \rangle$
10. $D_{30} = \langle s,t | s^{15} = t^2 = (st)^2 \rangle$
11. $D_3 \times D_5 = \langle a,b,s,t | a^3 = b^2 = s^5 = t^2 = (ab)^2 = (st)^2 = [a,s] = [b,s] = [a,t] = [b,t] = 1 \rangle$
12. $D_3 \times C_{10} = \langle s,t,u | s^3 = t^2 = u^{10} = (st)^2 = [s,u] = [t,u] = 1 \rangle$
13. $C_6 \times D_5 = \langle s,t,u | s^5 = t^2 = u^6 = (st)^2 = [s,u] = [t,u] = 1 \rangle$

Groups of order 72

There are 50 groups of order 72. However, we show the only groups that are interesting for our case.

1. $S_4 \times C_3 = \langle t,a,b | t^2 = a^3 = b^3 = (ta)^4 = [t,b] = [a,b] = 1 \rangle$
2. $(C_3 \times C_3) \rtimes_8 C_8 = \langle a,b,t | a^3 = b^3 = [a,b] = t^8 = 1, t^7at = b, t^7bt = ab \rangle$
3. $\big((C_3 \times C_3) \rtimes_1 C_4\big) \times C_2 = \langle a,b,t,s | a^3 = b^3 = [a,b] = t^4 = s^2 = [a,s] = [b,s] = [t,s] = 1, t^3at = a^2, t^3bt = b^2 \rangle$
4. $\big((C_3 \times C_3) \rtimes_2 C_4\big) \times C_2 = \langle a,b,t,s | a^3 = b^3 = [a,b] = t^4 = s^2 = [a,s] = [b,s] = [t,s] = 1, t^3at = b, t^3bt = a^2 \rangle$
5. $(C_3 \times C_3) \rtimes_1 D_4 = \langle a,b,s,t | a^3 = b^3 = [a,b] = t^4 = s^2 = (st)^2 = [s,a] = [s,b] = 1, t^3at = b, t^3bt = a \rangle = T \rtimes C_6$
6. $(C_3 \times C_3) \rtimes_2 D_4 = \langle a,b,s,t | a^3 = b^3 = [a,b] = t^4 = s^2 = (st)^2 = (sa)^2 = [s,b] = 1, t^3at = b, t^3bt = a^2 \rangle$
7. $(C_3 \times C_3) \rtimes_1 Q = \langle a,b,s,t | a^3 = b^3 = s^4 = t^4 = [a,b] = [b,t] = [a,s] = (ts)^2 = 1, t^2 = s^2, t^{-1}at = a^{-1}, s^{-1}bs = b^{-1} \rangle$
8. $(C_3 \times C_3) \rtimes_2 Q = \langle a,b,s,t | a^3 = b^3 = s^4 = t^4 = [a,b] = [a,s] = [b,t] = 1, t^2 = s^2, s^{-1}ts = t^{-1}, s^{-1}bs = b^{-1}, t^{-1}at = a^{-1} \rangle$
9. $(C_3 \times C_3) \rtimes_3 Q = \langle a,b,s,t | a^3 = b^3 = s^4 = t^4 = [a,b] = [a,b] = [b,t] = [a,s] = 1, t^2 = s^2, s^{-1}ts = t^{-1}, s^{-1}bs = b^{-1} \rangle$
10. $(C_3 \times C_3) \rtimes_4 Q = \langle a,b,s,t | a^3 = b^3 = s^4 = t^4 = [a,b] = [a,t] = [b,t] = 1, t^2 = s^2, s^{-1}bs = b^{-1}, s^{-1}as = a^{-1} \rangle$

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